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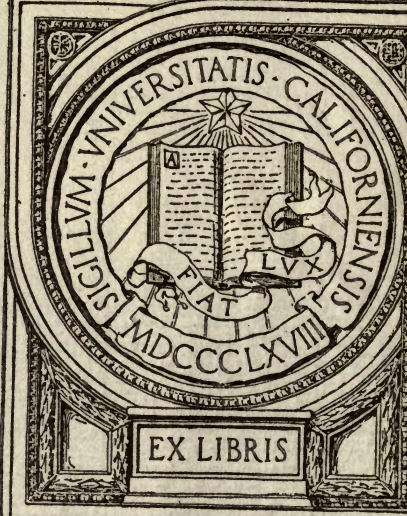
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THE TRANSITION CURVE

OR

CURVE OF ADJUSTMENT

AS APPLIED TO THE ALIGNMENT OF RAILROADS

BY THE METHOD OF RECTANGULAR CO-ORDINATES AND BY DEFLECTION ANGLES

(OR POLAR CO-ORDINATES)

BY

N. B. KELLOGG

M. AM. SOC. C. E.

THIRD EDITION

NEW YORK

McGRAW PUBLISHING COMPANY

1907

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replacement

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BY
N. B. KELLOGG

Stanhope Press
F. H. GILSON COMPANY
BOSTON U.S.A.

PREFACE

ERRATA

Page 8, Equation (17), read: $\frac{L^2}{2P}$ when $\frac{1}{r} = 0$.

Page 22, Line 5, read: With instrument at A, $D\Delta = 0$, since $D = 0$.

Page 23, Second line above footnote: change factor to term.

Page 37, Last line of Problem III, Case 2, read: as in the first part of this problem.

Page 46, Table, 5th column, near bottom: change 7.35 to 6.62.

Page 46, Table, 10th column, middle: change 1.07 to 1.57.

Page 65, Table, 7th column, top: change 08.5′ to 07.5′.

Page 69, near bottom of page, read: External Sec. a = Sec. $a - 1$.

Nordling equations, the formulæ (42–46) were obtained by the writer during the summer of 1884, as were also § (2–9) inclusive, by an independent process, resulting in the verification of the formulæ invented by Froude in 1841 and published in 1861. Hence, it is thought that

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PREFACE

WHILE the theory of the transition curve as here developed is based upon the methods given by M. Nordling (see "Annales des Ponts et Chaussées," 1867), the equations for both the cubic parabola and the spiral appear in Professor Rankine's "Civil Engineering" (Ed. 1863), and are accredited to Dr. William Froude, undoubtedly ante-dating those of Nordling.

Translations of a portion of Nordling's demonstration have appeared from time to time, but only so far as related to connecting a straight line with a circular curve. That portion relating to connecting circular curves of different radii, by means of the cubic parabola, has not appeared in the form given by Nordling, so far as I am aware. The formulæ deduced in the latter case are of general application and equally true for connecting curve with curve, or curve with tangent, when proper values are introduced into the equations.

That of joining a tangent with a circular curve, by means of the transition curve, is a special case where one of the radii becomes infinitely great. Some of the recent spirals, adopted as curves of adjustment in railroad location, easily develop from the equations of the cubic parabola by making the proper substitutions in them.

Following the supposition indicated by M. Nordling, i.e., regarding $x = L$ and substituting L for x in the Nordling equations, the formulæ (42–46) were obtained by the writer during the summer of 1884, as were also § (2–9) inclusive, by an independent process, resulting in the verification of the formulæ invented by Froude in 1841 and published in 1861. Hence, it is thought that

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the curve herein presented may, not improperly, be called Froude's Spiral. Since the spiral, as an adjustment curve, is free from some defects noticeable in the cubic parabola (appearing in the edition of 1899), the latter is omitted from the present edition.

The Compound Circular Transition Curve,* known also as the "Railroad Spiral" (an approximation to the Froude Spiral), though giving good results as an adjustment curve, is defective in not being expressed by a formula following a law of uniform change of curvature for consecutive points throughout its entire length. A chapter on the compound circular curve was prepared simultaneously with that of the true spiral and the cubic parabola, but was abandoned for the reason that the deduced formulæ were found to be not easily transposed or modified for arbitrary values of the offset distance and for the computation of fractional chords.

Few engineers, at the present time, will locate an important railway line without the use of some good curve of adjustment.

A few simple methods of the calculus are used to derive the formulæ, but a knowledge of only the ordinary processes of algebra and trigonometry is required for

* If a *compound circular transition curve* and a *spiral* be run for the same offset, the origin of the compound circular curve will be fonnd one-half chord length *in advance* of that of the true spiral and the terminus of the compound circular curve one-half chord length *back* of that of the spiral, nearly, i.e., the *ends* of the chords of the spiral correspond closely to the *middle points* of the several *arcs* of the compound circular transition curve. For instance, a compound circular curve of *nine* equal chords will have, nearly, the same offset (*f*) as a spiral of *ten* equal chords of the same length, each curve terminating in the same radius of curvature as the main curve; the total curvatures, however, will differ, since the spiral is tangent to the main curve nearly one-half chord length *beyond* where the compound circular curve would become tangent to the main curve.

their application. It is seldom necessary to use more than two terms in the formulæ involving series.

The *larger type* may be read independently of the smaller type. The latter may be used when the curve is extended beyond ordinary limits and greater accuracy in results is sought.

For field practice, formulæ (42–52), (60–61), are the ones with which, taken with the tables, one should become familiar; and these formulæ may be regarded as a summary of the essential part of the book.*

The "run off" is understood to be equal in length to that of the transition curve and coincident with it.

The transition or adjustment curve can rarely be applied to switches or turnouts to advantage, except where high speeds are maintained.

The more important formulæ are in **full-faced type.**

I am indebted to Professor H. I. Randall, C.E., of the University of California, and also to Mr. D. E. Hughes, C.E., for valuable suggestions. The latter is the author of an excellent paper on this subject.

N. B. K.

San Francisco, January, 1904.

* By making all terms containing the factors m or f, *representing spiral offsets* in Probs. (I–V) *equal zero*, the spiral disappears from the formula and it stands for circular curves simple or compound as the case may be.

Except in a few cases, Greek letters have been used to designate angles, small Roman italics for lines and Roman capitals for points.

S. F. 1907.

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THE TRANSITION CURVE

§ 1. The true transition curve is one of which the radius of curvature, at its origin, is infinitely great; and at any other of its points, the radius of curvature is inversely proportional to the distance of the point, measured on the curve, from the origin; the product of the radius and distance being a constant for all points of the curve.

The object of introducing the transition curve between circular curves of different radii, or between a tangent and circular curve, as applied to the alignment of railroads, is to give centrifugal force an appreciable time to develop, from that due to one given radius of curvature to that of another, in a moving body passing from a circular path of one rate of curvature to that of another rate of curvature; and to develop simultaneously a force equal and opposed to the centrifugal force, neutralizing it at every point of the curve. If one of the given radii is made infinitely great, its curve becomes a tangent; and centrifugal force develops gradually from zero to that due to the other given radius of curvature, in the time it takes to traverse the transition curve; thus avoiding instantaneous development or "shock."

The opposing force is the horizontal component of a force due to gravity developed by inclining the vertical axis, passing through the center of gravity of the moving body, from a normal to the plane of rotation and towards the center of rotation.

From mechanics the expression for centrifugal force

f of a weight w moving in a circular path with a radius r and a velocity v is:

$$f = \frac{wv^2}{32.2r}.$$

The opposing horizontal force due to gravity is (Fig. 1):

$$w_1 = \frac{we}{g},$$

in which w is the weight of the moving body, g the width of the path or gauge of the track, and e the inclination of the path, or "cant," towards the center of rotation, in a distance g.

Since by the hypothesis the opposing forces equal each other in intensity,

$$f = w_1$$

or

$$\frac{we}{g} = \frac{wv^2}{32.2r}, \text{ solving for } e$$

$$e = \frac{gv^2}{32.2r}, \text{ nearly.*}$$

---

* More correctly (Fig. 1) $e = \frac{hv^2}{\sqrt{(32.2)^2 r^2 + v^4}}$

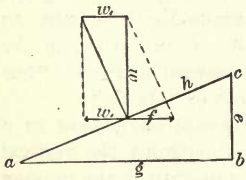

Fig. 1.

in which h = gauge of track.

The resultant of w and w_1 due to the effect of gravity = $\sqrt{w^2 + w_1^2}$ which is the pressure normal to the plane of the path when centrifugal force is developed.

It may not be out of place to remark here, that it is the usual custom in fixing superelevation, to depress the inner rail below the grade of the center line of the track $\frac{1}{2}e$ and elevate the outer rail above the grade line $\frac{1}{2}e$, thus maintaining the center line at grade.

If i denote the distance, measured on the curve, required to change the inclination or "cant" of the path one unit, $\frac{1}{i}$ will denote the change of cant in one unit of distance. Therefore, in a distance $OB_{,} = L_{,}$ the cant, or superelevation, will be:

$$\frac{L_{,}}{i} = e = \frac{gv^2}{32.2r_{,}} \tag{1}$$

from which

$$L_{,}r_{,} = \frac{gv^2 i}{32.2} \tag{2}$$

The second member of Eq. (2) is a constant, since all its factors are assumed constant, and represented by P, whence, **the fundamental equation of the curve:**

$$L_{,}r_{,} = P, \tag{3}*$$

which is the equation of a spiral, in which $OB_{,} = L_{,}$, Fig. 2, represents the required length of the transition curve, $DB_{,} = r$, the radius of the curvature common to the transition curve and the given circular curve where these curves become tangent each other. (See Fig. 2.)

GENERAL FORMULA.

§ 2. If in the figure we let $OB_{,}$ be denoted by $L_{,}$ and $OB_{,,}$ by $L_{,,}$, then

$$OB_{,} = L_{,} = \frac{P}{r_{,}};\ OB_{,,} = L_{,,} = \frac{P}{r_{,,}};$$

subtracting, letting $L_{,,} - L_{,} = L$, we have the *General Formula* for compound curves.

---

* Froude's Eq. (3), is in the form of $L = ei$, which is called by Rankine "The Curve of Adjustment." Any particular spiral is designated by the numerical value assigned to its constant P.

$$B_{,}B_{,,} = L_{,,} - L_{,} = L = \left(\frac{P}{r_{,,}} - \frac{P}{r_{,}}\right) = P\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right) \quad (4)$$

which is the ideal equation of all Transition Curves, $B_{,}$ being the first and $B_{,,}$ the second point of compound.

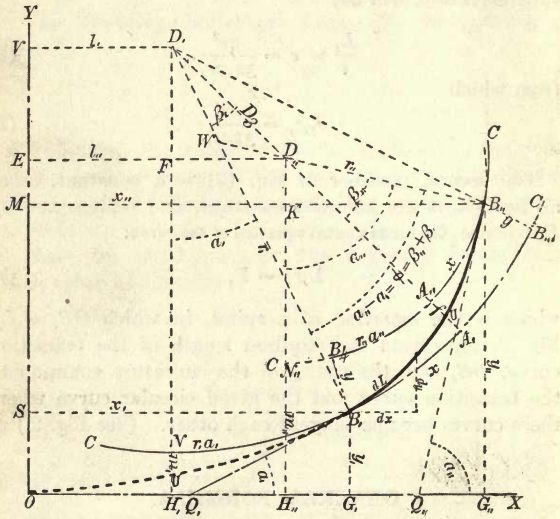

Fig. 2.

It fulfills every condition of theory and offers no unusual difficulties in application.

When the length L of the spiral and the radii $r_{,,}$ and $r_{,}$ are assumed.

The constant $P = \left(\frac{Lr_{,,}r_{,}}{r_{,} - r_{,,}}\right)$

or

$$P = \frac{L}{\frac{1}{r_{,,}} - \frac{1}{r_{,}}} \quad (5)$$

§ 3. In general let $L = \frac{P}{r} = Pr^{-1}$; differentiating $dL = Pr^{-2}dr$. We have from the calculus: $dL_{,} = rda$, whence

$$rda = Pr^{-2}dr,\ da = \frac{Pr^{-2}dr.}{r} = Pr^{-3}dr;\ \text{or}$$

$$a = \int Pr^{-3}dr = \frac{Pr^{-2}}{2} = \frac{P}{2r^2};$$

hence for any two arcs $a_{,,}$ and $a_{,}$ at radius = 1, we have:

$$a_{,,} = \frac{P}{2r_{,,}^{\ 2}},\ a_{,} = \frac{P}{2r_{,}^{\ 2}}; \tag{6}$$

$$a_{,,} - a_{,} = \frac{P}{2r_{,,}^{\ 2}} - \frac{P}{2r_{,}^{\ 2}} = \frac{P}{2}\left(\frac{1}{r_{,,}^{\ 2}} - \frac{1}{r_{,}^{\ 2}}\right),$$

or

$$a_{,,} - a = \frac{P}{2}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)\left(\frac{1}{r_{,,}} + \frac{1}{r_{,}}\right).$$

$$\frac{L}{2} = \frac{P}{2}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right); \tag{7}$$

substituting in value for $a_{,,} - a_{,}$ we have for the **Central Angle for Compound Curves expressed in arc** at radius = 1.

$$* \ a_{,,} - a_{,} = \frac{L}{2}\left(\frac{1}{r_{,,}} \pm \frac{1}{r_{,}}\right) = \phi = \beta_{,,} \pm \beta_{,} \tag{8}$$

---

* The values of $\beta_{,}$ $\beta_{,,}$ $a_{,}$ $a_{,,}$ and ϕ are expressed in arc at radius = 1. The degrees in an arc of a circle which is equal to the radius in length = 57.295. The arc of 1 degree at radius = 1 is .01745+. Hence the expression for any number of degrees in a given circular arc is: $A^{\circ} = \frac{\text{arc } A}{r\ 01745\ +}$.

in which $$\phi = \beta_{//} \pm \beta_{/} \qquad (9)$$

is the angle subtended by the arc L of the *spiral* as well as the sum of the circular arcs $A_{/}B_{/} + A_{//}B_{//}$

the + sign being used when the directions of curvature are similar;

the − sign when the *directions of curvature are reversed.*

EXAMPLE No. 1. Given $L = 150$; $r_{//} = 818.8$; $r = 2865$; to find arc $(\alpha_{//} - \alpha_{/}) = \phi = \beta_{//} + \beta_{/} = \frac{L}{2}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)$,

$$\text{arc } (\alpha_{//} - \alpha_{/}) = \phi = \frac{150}{2}\left(\frac{1}{818.8} + \frac{1}{2865}\right) = .11775$$

$$= \text{arc of } 6° 45'.$$

If in equation (8) we make $r_{/} = \infty$, and $\alpha_{/} = 0$, then

$$\alpha_{//} = \frac{L_{//}}{2r_{//}}; \quad \alpha_{//} = \qquad (10)$$

the angle, expressed in arc, subtended by spiral $OB_{//}$.

§ 4. DIFFERENCE IN LENGTH OF THE SPIRAL AND THE SUM OF THE CIRCULAR ARCS SUBTENDING THE SAME ANGLE.

The rectangular co-ordinates of the center of the circle with radius $r_{//}$, as $D_{//}$, referred to the origin O are: $OH_{/} = l_{/}$ and $H_{/}D_{/} = k_{/}$, and of $D_{//}$ $OH_{//} = l_{//}$, $H_{//}D_{//} = k_{//}$; whence

$$\frac{l_{//} - l_{/}}{k_{/} - k_{//}} = \tan(\alpha_{/} + \beta_{/}); \qquad (11)$$

$$l_{//} = x_{//} - r_{//}\sin\alpha_{//}, \qquad l_{/} = x_{/} - r_{/}\sin\alpha_{/};$$

$$k_{//} = r_{//} + m_{//}, \qquad k_{/} = r_{/} + m_{/};$$

whence

$$\tan(\alpha_{/} + \beta_{/}) = \frac{(x_{//} - r_{//}\sin\alpha_{//}) - (x_{/} - r_{/}\sin\alpha_{/})}{(r_{/} + m_{/}) - (r_{//} + m_{//})}, \qquad (12)$$

or if we have the value of f

$$\cos(\alpha_{/} + \beta_{/}) = \frac{(r_{/} + m_{/}) - (r_{//} + m_{//})}{(r_{/} - (r_{//} + f)}, \qquad (13)$$

from either of these equations

$$\beta_{//}{}^{\circ} = \alpha_{//}{}^{\circ} - (\alpha_{/}{}^{\circ} + \beta_{/}{}^{\circ}), \tag{14}$$

with $\beta_{//}{}^{\circ}$ reduced to arc, then

$$r_{//}\beta_{//} = A_{//}B_{//}$$

and

$$\beta_{/} = \alpha_{//} - (\alpha_{/} + \beta_{//}),\ r_{/}\beta_{/} = A_{/}B_{/}; \tag{15}$$

denoting the difference between L and $A_{//}B_{//} + A_{/}B_{/}$ by "d" we have

$$d = L - (A_{//}B_{//} + A_{/}B_{/}) = L - (r_{//}\beta_{//} + r_{/}\beta)$$

Placing

$$(r_{//}\beta_{//} + r_{/}\beta_{/}) = Lc,$$

we have

$$\text{"d"} = \text{L} - \text{Lc}. \tag{16}$$

§ 5. TRANSFORMATION TO RECTANGULAR CO-ORDINATES.

By the Calculus $dy = dL \sin \phi$

$$dx = dL \cos \phi;$$

in which ϕ = any angle. By trigonometry,

$$\sin \phi = \frac{\phi}{1} - \frac{\phi^3}{1.2.3} + \frac{\phi^5}{1.2.3.4.5} - \text{etc.} \ldots\ldots \frac{\phi^n}{1.2.3..n};\ *$$

$$\cos \phi = 1 - \frac{\phi^2}{1.2} + \frac{\phi^4}{1.2.3.4} - \text{etc.} \ldots\ldots \frac{\phi_m}{1.2.3..m};$$

---

* This and the succeeding formula are for expressing the trigonometrical function of an arc in terms of the arc itself (see Chauvenet's Trigonometry, Chapter XIII, 1867).

m and n = positive whole numbers. m = even, n = odd.

in which m may have any value from zero to infinity, n may have any value from one to infinity, and ϕ any value from zero to infinity. Substituting in the equations for sin ϕ and cos ϕ,

$$\phi = \frac{L}{2}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right) = \frac{L^2}{2P} \tag{17}$$

we have

$$\frac{dy}{dL} = \sin\phi = \sin(\beta_{//} + \beta_{/}) = \sin B_{/}WB_{//} =$$

$$\frac{L^2}{2P}\left(1 - \frac{L^4}{24P^2} + \frac{L^8}{1920P^4} - \text{etc.}\right) \tag{17a}$$

or $$\sin\phi = \frac{L}{2}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)\left[1 - \frac{L^2}{24}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)^2 + \text{etc.}\right] \tag{17b}$$

EXAMPLE No. 2. Given $L = 150$; $r_{//} = 818.8$; $r_{/} = 2865$; to find ϕ.

$$\sin\phi = \sin(\beta + \beta) = \frac{L}{2}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)$$

$$\left[1 - \frac{L^2}{24}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)^2 + \text{etc.}\right]$$

$$\sin\phi = \sin(\beta_{//} + \beta_{/}) = \frac{150}{2}\left(\frac{1}{818.8} + \frac{1}{2865}\right)\left[1 - \frac{(150)^2}{24}\left(\frac{1}{818.8} + \frac{1}{2865}\right)\right.$$

$$\sin\phi = \sin(\beta_{//} + \beta_{/}) = .11775 \times .9977 = .11748.$$

If we make $r_{/} = \infty$, $\frac{1}{r_{/}} = 0$;

we have, with origin at O, $\phi = a_{//}$; and $L = L_{//}$ (17b) becomes

$$\sin a_{//} = \frac{L_{//}}{2r_{//}}\left(1 - \frac{L_{//}^2}{24r_{//}^2} + \frac{L_{//}^4}{1920\,r_{//}^4} - \text{etc.}\right) = \sin B_{//}D_{//}N_{//} \tag{17c}$$

$$dy = dL\left(\frac{L^2}{2P} - \frac{L^6}{48P^3} + \frac{L^{10}}{3840P^5} - \text{etc.} \ldots \frac{L^n}{NP^{\frac{n}{2}}}\right); \qquad (18)$$

integrating,

$$y = \frac{L^3}{6P}\left(1 - \frac{L^4}{56P^2} + \text{etc.} \ldots \frac{L_{//}^{n-2}}{(n+1)\,NP^{\frac{n}{2}-1}}\right); \qquad (19)$$

or, since $P = \dfrac{L}{\frac{1}{r_{//}} - \frac{1}{r_{/}}}$, with its origin at $B_{/}$, the ordinate of any point of the spiral, as

$$B_{//}B_{///} = y = \frac{L^2}{6}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)\left[1 - \frac{L^2}{56}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2 + \text{etc.}\right]. \qquad (20)$$

Example 3. Given $r_{/} = 2865$; $r_{//} = 818.8$; $L = 150$, to find y by:

$$y = \frac{L^2}{6}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)\left[1 - \frac{L^2}{56}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2 + \text{etc.}\right] =$$

$$\frac{(150)^2}{6}(.00122 - .00035)\left[1 - \frac{(150)^2}{56}(.00122 - .00035)^2 + \text{etc.}\right]$$

$$y = \frac{22500 \times .00087}{6}\left[1 - 2500\,(.00087)_2 + \text{etc.}\right. = 3.2625\,[.9997]$$

$$= 3.26225.$$

Or in terms of ϕ, since $L = 2r\phi$, and $\frac{1}{r} = \left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)$

$$y = \frac{2}{3}r_{/}\phi^2\left(1 - \frac{\phi^2}{14} + \right) = \frac{\frac{2\phi^2}{3}}{\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)}\left[1 - \frac{\phi}{14} + \text{etc.}\right]. \qquad (20a)$$

If we make $r_{,} = \infty$, $\frac{1}{r_{,}} = 0$, $L = L_{,,}$ the ordinate of any point of the spiral, as

$$B_{,,}G_{,,} = y_{,,} = \frac{L_{,,}^2}{6r_{,,}}\left(1 - \frac{L_{,,}^2}{56r_{,,}^2} + \text{etc.}\right) \quad \text{with origin at O} \qquad (21)$$

and $OG_{,,}$ as abscissa.

Similarly,

$$\frac{dx}{dL} = \cos\phi = \left(1 - \frac{L^4}{8P^2} + \frac{L^8}{384P^4} - \text{etc.}\right) \qquad (22a)$$

$$dx = dL\left(1 - \frac{L^4}{8P^2} + \frac{L^8}{384P^4} - \frac{L^{12}}{46080P^6} + \text{etc.} \ldots \frac{L^m}{MP^m}\right)$$

integrating,

$$x = L\left(1 - \frac{L^4}{40P^2} + \frac{L^8}{3456P^4} - \frac{L^{12}}{599040P^6} + \text{etc.} \ldots \frac{L^m}{(m+1)\,MP^m}\right); \qquad (23)$$

With origin at $B_{,}$ then the abscissa of any point of the spiral, as

$$B_{,}B_{,,} = x = L\left[1 - \frac{L^2}{40}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)^2 + \frac{L^4}{3456}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)^4 - \text{etc.}\right. \qquad (24)$$

or in terms of ϕ, since $L = 2r\phi$, (25)

$$x = 2r\phi\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216}\right) = \frac{2\phi}{\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)}\left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216} - \text{etc.}\right)$$

x is laid off on the arc of the circle $A_{,}B_{,}$ or $A_{,,}B_{,,}$ (with radii $r_{,}$ or $r_{,,}$) as axis of X with $B_{,}$ or $B_{,,}$ as origin, x being the abscissa of y which is laid off normal, or radial, to the arc $A_{,}B_{,}$ or $A_{,,}B_{,,}$.

EXAMPLE 4.

$$x = L\left[1 - \frac{L^2}{40}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2 \pm \text{etc. (other terms)}\right]$$

$$= 150\left(1 - \frac{22500 \times .0000007569 +}{40} \pm\right);$$

$$x = 150\ (.9996 \pm) = 149.94.$$

With origin at O, any abscissa, as

$$OG_{//} = x_{//} = L_{//}\left(1 - \frac{L_{//}^2}{40\,r_{//}^2} + \frac{L_{//}^4}{3456 r_{//}^4} - \frac{L_{//}^6}{599040\,r_{//}^6} + \text{etc.}\right); \quad (26)$$

also when $\phi = \alpha_{//}$

$$x_{//} = 2r_{//}\alpha_{//}\left(1 - \frac{\alpha_{//}^2}{10} + \frac{\alpha_{//}^4}{216} - \text{etc.}\right) \quad (26a)$$

§ 6. RECTANGULAR CO-ORDINATES AT THE OFFSET DISTANCE.

1°. To find the abscissa and length of the curve.

Let $y_{m//}$ be the ordinate, $x_{m//}$ the abscissa, and $L_{m//}$ the length of the curve corresponding to $x_{m//}y_{m//}$; then a very nearly correct value of $y_{m//}$ for central angles up to 50 degrees may be obtained as follows:

$OH_{//} = x_{m//} = x_{//} - r_{//} \sin \phi$: whence, with origin at O, (27)

by Eq. * page 7 and (27).

$$OH_{//} = x_{m//} = r_{//}\phi\left(1 - \frac{\phi^2}{30} + \frac{\phi^4}{1080} - \ldots\right) \text{ or} \quad (27a)$$

in terms of $r_{//}$ and $L_{//}$

$$OH_{//} = x_{m//} = \frac{L_{//}}{2}\left(1 - \frac{L_{//}^2}{120\,r_{//}^2} + \frac{L_{//}^4}{17280\,r_{//}^4} \ldots\right) \quad (28)$$

and when the origin is at $B_{/}$

$$B_{/}A_{/} = x_{/} = \frac{L}{2}\left[1 - \frac{L^2}{120}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2 + \frac{L^4}{17280}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^4 - \right] \quad (28a)$$

in which $L = L_{//} - L_{/} = B_{/}B_{//}$.

A close approximation to L_f or L_m is L_a which is determined by making

$$L_a = \left(\frac{\frac{L}{2} + x_f}{2}\right) - \left(\frac{\frac{L}{2} - x_f}{3}\right) = \frac{L + 10x_f}{12} \qquad (29)$$

whence $$r_a = \frac{P}{L_a},$$

with origin at B_t, then

$$* B_tU = L_f = \frac{x_f}{1 - \frac{L_a^2}{40}\left(\frac{1}{r_a} - \frac{1}{r_t}\right)^2} \text{ and } r_f = \frac{P}{L_f + L_t}; \; r_m = \frac{P}{L_m} \qquad (30)$$

We may write L_m for L_f and x_m for x_f, r_m for r_a, $\frac{1}{r_t} = 0$, in equation (30) when the origin is at O; then

$$L_m = \frac{x_m}{1 - \frac{L_a^2}{40 r_a^2}} \qquad (30a)$$

2°. To find the ordinate of a transition curve measured on the offset $m_{//}$ (or f):

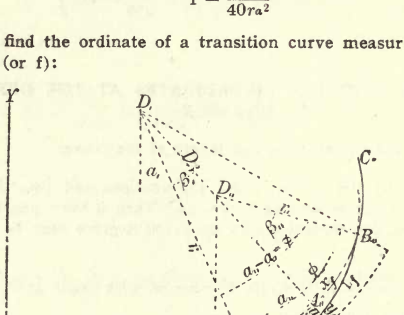

Fig. 2a.

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* After solving (Eq. 30) for L_f, compute r_f, give L_a and r_a, in the denominator, the value just found for L_f and r_f, and solve again for L_f till the Eq. $x_f = L_f\left[1 - \frac{L_f^2}{40}\left(\frac{1}{r_f} - \frac{1}{r_t}\right)^2\right.$ etc., is true, x_f having a fixed value from Eqs. (27–28a).

The method given in Equation (47) for finding the ordinate y f is approximate and sufficiently close within the limits of the tables. For a large central angle or great length of spiral a closer approximation is necessary.

Writing $y_{m''}$ for y, when origin is at O, we have by (19):

$$y_{m''} = \frac{L_{m''}^3}{6P}\left(1 - \frac{L_{m''}^4}{56P^2} + \text{etc.}\right) = \frac{L_{m''}^2}{6r_{m''}}\left(1 - \frac{L_{m''}^2}{56r_{m''}^2} + \text{etc.}\right); \quad (31)$$

with origin at B, then $B'U = Lf = P\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$, and

$$A\,U = y_f = \frac{L_f^2}{6}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)\left[1 - \frac{L_f^2}{56}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)^2 + \text{etc.}\right] \quad (32)$$

$$\beta_i = \frac{xf}{r_i}; \quad \beta_{i''} = \phi - \beta_i = (\alpha_{i''} - \alpha_i) - \beta_i.$$

This method of obtaining $\beta_{i''}$ and β_i may be used instead of Eq. 6 or 8.

§ 7. OFFSET DISTANCE.

In general, by Fig. (2), the offset distance.

$$f = y - R \operatorname{ver} \phi = y - R\,(1 - \cos \phi), \quad (33)$$

in which R depends on $\frac{1}{r_{i''}} - \frac{1}{r_i}$ for its value, substituting for $\cos \phi$ its value $\left(\text{remembering that } R = \frac{P}{L}\right)$, and reducing, since

$$R\,(1 - \cos \phi) = R\left[1 - \left(1 - \frac{L^4}{8P^2} + \frac{L^8}{384P^4} - \text{etc.}\right)\right],$$

$$f = y - R \operatorname{ver} \phi = L\left[\left(\frac{L^2}{6P} - \frac{L^6}{336P^3} + \text{etc.}\right) - \left(\frac{L^2}{8P} - \frac{L^6}{384P^3} + \text{etc.}\right)\right]; \quad (34)$$

$$f = L\left(\frac{L^2}{24P} - \frac{L^6}{2688P^3} + \text{etc.}\right) = \frac{L}{P}\left(\frac{L^2}{24} - \frac{L^6}{2688P^2} + \text{etc.}\right); \quad (35)$$

$$\text{since } \frac{L}{P} = \left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right), \text{ we have for} \quad (36)$$

the offset with origin at $B_{/}$:

$$A_{/}A_{//} = f = \frac{L^2}{24}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)\left[1 - \frac{L^2}{112}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2 + \text{etc.}\right] \quad (37)$$

EXAMPLE 5. Given $r_{//} = 818.8$; $r_{/} = 2865$; $L = 150$; to find the offset f. First, when the circular curves $r_{//}$ and $r_{/}$ turn in the same direction, by (125):

$$f = \frac{L^2}{24}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)\left[1 - \frac{L^2}{112}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)^2\right] =$$

$$\frac{(150)^2}{24} \times .00087\left[1 - \frac{(150)_2\,(.00087)^2}{112}\right] = .815.$$

With origin at O, when ϕ becomes $\alpha_{//}$, L becomes $L_{//}$ and f becomes $m_{//}$, then

$$* \quad N_{//}H_{//} = m_{//} = \frac{L_{//}^2}{24r_{//}}\left(1 - \frac{L_{//}^2}{112r_{//}^2} + \text{etc.}\right) \quad (38)$$

Second, when the circular curves turn in opposite directions or are reversed.

$$m_{//} + m_{/} = \frac{1}{24}\left[\left(\frac{L_{//}^2}{r_{//}} + \frac{L_{/}^2}{r_{/}}\right) - \frac{1}{112}\left(\frac{L_{//}^4}{r_{//}^3} + \frac{L_{/}^4}{r_{/}^3}\right)\right] = \quad (39)$$

If $L_{//} = L_{/}$, then

$$\dagger \quad m_{//} + m_{/} = \frac{L_{//}^2}{24}\left(\frac{1}{r_{//}} + \frac{1}{r_{/}}\right)\left[1 - \frac{L_{//}^2}{112}\left(\frac{1}{r_{//}^2} - \frac{1}{r_{//}r_{/}} + \frac{1}{r_{/}^2}\right) + \right] \quad (40)$$

For the distance between the centers of the two curves, turning in the same direction, we have:

$$r_{/} - (r_{//} + f) = D_{/}D_{//} = r_{/} - \left[r_{//} + \frac{L^2}{24}\left(\frac{1}{r_{//}} - \frac{1}{r_{/}}\right)(1 - \text{etc.})\right]. \quad (41)$$

---

* For the constant P, when $OB_{/} = B_{/}B_{//}$ or $L_{/} = L$, then $H_{/}U = A_{/}U$, $m_{/} = f$, and $B_{/}G_{/} = B_{//}B_{///}$, or $y_{/} = y = B_{/}B_4$ (nearly).

† In this case there will be two values of P except when $r_{//} = r_{/}$.

EXAMPLE 6. Given $L = 150$; $r_{\prime} = 2865$; $r_{\prime\prime} = 818.8$; to find the distance between the centers of curves with radii $r_{\prime\prime}$ and $r_{\prime}$. First, if curves turn in the same direction (Fig. 2):

$$D_{\prime\prime}D_{\prime} = r_{\prime} - \left[r_{\prime\prime} + \frac{L^2}{24}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right)(1 - \text{etc.})\right].$$

$$D_{\prime\prime}D_{\prime} = r_{\prime} - (r_{\prime\prime} + f) = 2865 - \left[818.8 + \frac{(150)^2}{24} + .00087\right];$$

$$D_{\prime\prime}D_{\prime} = 2865 - [818.8 + .815] = 2865 - 819.615 = 2045.385.$$

Second, if curves are reversed (or turn in a contrary direction) (Fig. 9), then:

$$D_{\prime\prime}D_{\prime\prime\prime} = \left[r_{\prime} + \left(r_{\prime\prime} + \frac{1}{24}\left(\left(\frac{L_{\prime\prime}{}^2}{r_{\prime\prime}} + \frac{L_{\prime}{}^2}{r_{\prime}}\right) - \frac{1}{112}\left(\frac{L_{\prime\prime}{}^4}{r_{\prime\prime}{}^3} + \frac{L^4}{r_{\prime}{}^3}\right)\right)\right)\right] \quad (41a)$$

$$\times \sec K_{\prime\prime}D_{\prime}D_{\prime\prime\prime} = [r_{\prime} + (r_{\prime\prime} + m_{\prime\prime} + m_{\prime})] \sec K_{\prime\prime}D_{\prime\prime}D_{\prime\prime\prime};$$

$$\tan K_{\prime\prime}D_{\prime}D_{\prime\prime\prime} = \frac{xm_{\prime\prime} + xm_{\prime}}{r_{\prime} + r_{\prime\prime} + m_{\prime\prime} + m_{\prime}}. \quad (41b)$$

PRACTICAL FORMULÆ.

§ 8. If, in the foregoing values of y, x, $\frac{dy}{dL}$, and f we omit terms in the bracket after the first or second, we have for central angles of 20°, or less, the following **practical formulæ for uniting circular curves of different radii by means of a spiral arc:**

$$B_{\prime\prime}B_{\prime\prime\prime} = y = \frac{L^2}{6}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right); \quad (42)$$

See Eq. (32)

$$A_{\prime}U = y_f = \frac{L_f{}^2}{6}\left(\frac{1}{r_f} - \frac{1}{r_{\prime}}\right)\left[1 - \frac{L_f{}^2}{56}\left(\frac{1}{r_f} - \frac{1}{r_{\prime}}\right)^2 + \text{etc.}\right. \quad (42a)$$

$$B_{\prime}B_{\prime\prime\prime} = x = L\left[1 - \frac{L^2}{40}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right)^2 + \right. \quad (43)$$

See Eq. (28a)

$$B_{\prime}A_{\prime} = x_f = \frac{L}{2}\left[1 - \frac{L^2}{120}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right)^2 + \ldots\right] \quad (43a)$$

---

* When $O_{\prime}O_{\prime\prime} = 0$.

$$B_{,}B_{,,} = L = x\left(1 + \frac{x^4}{40P^2}\right); \text{ nearly} \qquad (44)$$

$$L_{,} = \frac{L + 10x_{,}}{12} \text{ see Eq. (29)} \qquad (44a)$$

$$\sin B_{,}WB_{,,} = \sin\phi = \sin(\beta_{,,} + \beta_{,}) = \qquad (44b)$$

$$\frac{L}{2}\left(\frac{1}{r_{,,}} + \frac{1}{r_{,}}\right)\left[1 - \frac{L^2}{24}\left(\frac{1}{r_{,,}} + \frac{1}{r_{,}}\right)^2 + \text{etc.}\right.$$

$$\text{arc } B_{,}WB_{,,} \text{ (at radius } = 1) = \frac{L}{2}\left(\frac{1}{r_{,,}} + \frac{1}{r_{,}}\right) = \text{arc } \phi: \qquad (44c)$$

$$\text{arc } \phi \times 57.3^\circ = \phi^\circ$$

$$A_{,}A_{,,} = f = \frac{L^2}{24}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right), \qquad (45)$$

from which latter equation we find:

$$\frac{B_{,}B_{,,}}{2} = \frac{L}{2} = \sqrt{6f \div \left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)} \qquad (46)$$

identical with Froude's curve of adjustment, as indicated by Professor Rankine (" Civil Engineering," Edition 1863).

$\frac{1}{2}f = \frac{L^2}{48}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right)$, and if in the value for y we write $\frac{1}{2}L$ for L and cube it, then

$$y = \frac{(\frac{1}{2}L)^3}{6P} = \frac{\frac{L^3}{8}}{6P} = \frac{L^3}{48P} = \frac{L^2}{48R} = \frac{L^2}{48}\left(\frac{1}{r_{,,}} - \frac{1}{r_{,}}\right). \qquad (47)$$

Hence the ordinate $y = \frac{1}{2}f$ is at the middle point of the transition curve.

For uniting a tangent with a circular curve by the use of the spiral,

we have $\frac{1}{r_{/}} = o$, in (42–47);

whence: by Fig. 2, for any distance on the spiral, as $L_{//}$, with radius $r_{//}$, the corresponding

$$\text{Ordinate } G_{//}B_{//} = y_{//} = \frac{L_{//}^{2}}{6r_{//}} \qquad (48)$$

$$\text{and the abscissa } OG_{//} = x_{//} = L_{//}\left[1 - \frac{L_{//}^{2}}{40r_{//}^{2}} + \right] \qquad (49)$$

The sine of central angle of the transition curve equals

$$\sin N_{//}D_{//}B_{//} = \sin \alpha_{//} = \frac{L_{//}}{2r_{//}}\left(1 - \frac{L_{//}^{2}}{24r_{//}^{2}}\right) \qquad (50)$$

$$\text{arc } N_{//}D_{//}B_{//} = \alpha_{//} = \frac{L_{//}}{2r_{//}}, \quad \text{arc } \alpha_{//} \times 57.3^{\circ} = \alpha_{//}^{\circ}$$

$$\text{the offset } N_{//}H_{//} = m_{//} = \frac{L_{//}^{2}}{24r_{//}} \qquad (51)$$

$$\tfrac{1}{2} \text{ the length of spiral} = \frac{L_{//}}{2} = \sqrt{6m_{//}r_{//}} \qquad (52)$$

$$y_{m_{//}} = \frac{L_{m_{//}}^{3}}{6P}\left(1 - \frac{L_{m_{//}}^{4}}{56P^{2}} + \text{etc.}\right) = \frac{L_{m_{//}}^{2}}{6r_{m_{//}}}\left(1 - \frac{L_{m_{//}}^{2}}{56r_{m_{//}}^{2}} + \text{etc.}\right); \qquad (52a)$$

$$OH_{//} = x_{m_{//}} = x_{//} - r_{//}\sin\alpha_{//}, \text{ or} \qquad (a)$$

$$OH_{//} = x_{m_{//}} = \frac{L_{//}}{2}\left(1 - \frac{L_{//}^{2}}{120r_{//}^{2}} + \frac{L_{//}^{4}}{17280r_{//}^{4}} \cdots\right) \qquad (52b)$$

$O_{/}d$ or $O_{/}I = t = x_{m} + (r_{//} + m)\tan\tfrac{1}{2}I$ (see Fig. 5) (b)

If $D_{//}$ and $D_{/}$ denote the "degrees" of the curves (determined by 100 feet of their length) corresponding to the radii, $r_{//}$ and $r_{/}$; then, since $L_{//}$ and $D_{//}$ and $L_{/}$ and $D_{/}$ are inverse functions of $r_{//}$ and $r_{/}$, we have

$$L_{//} - L_{/} = L = L_{//}\frac{D_{//} - D_{/}}{D_{//}} \text{ and } \frac{1}{r_{//}} - \frac{1}{r_{/}} = \frac{L_{//}}{P}\left(\frac{D_{//} - D_{/}}{D_{//}}\right)$$

TO LAY OUT THE TRANSITION CURVE BY DEFLECTIONS.

§ 9. If at the point $B_{'}$ (see Fig. 2) we imagine $r_{'}$ to increase until it becomes infinitely great, the curvature of $B_{'}A_{'} = 0$ and the arc $B_{'}A_{'}$ will be a straight line still preserving its tangency to the transition curve. The curvature at $B_{''}$ will diminish to the same extent, i.e., the *difference* between curvature at $B_{'}$ and $B_{''}$ will be the same as when $\beta_{'} = 0$, and $\beta_{''} = \frac{L^2}{2P}$. The ordinates x y can be computed and laid off from the new tangent as axis of abscissa with $B_{'}$ as origin, the same as if from O *.

If we now conceive this new axis of x to be curved to a radius $r_{'}$ the curvature of the transition curve at any point will be increased by the same amount and the ordinates may, without serious error, be laid off normal to the arc $B_{'}A_{'}$ and establish points of the transition curve. The same reasoning will apply if $r_{''} = \infty$ and $B_{''}A_{''}$ becomes tangent and values of x and y be laid off from it with $B_{''}$ as origin, except that the resulting transition curve would be convex to $B_{''}A_{''}$. The ordinates would, however, be equal to those of the corresponding distance from $B_{'}$. If $r_{''}$ now resume its original length the curvature of the transition curve at any point will equal that of the circular curve with radius $r_{''}$ *minus* the curvature it had in a contrary direction when $r_{''}$ was infinitely great and $B_{''}A_{''}$ a straight line. In determining $\beta_{''}$ and $r = \frac{P}{L}$, data may be taken from the tables.

Equations for x and y are equally true whether the origin be taken as O, $B_{'}$, or $B_{''}$.

x will be measured on $A_{'}B_{'}$, and y normal to $A_{'}B_{'}$.

To lay the transition curve out by the expressions

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* In Fig. 2, $O_{'}$ should be marked B. S. or E. S. and $B_{'}$ or $B_{''}$ marked B. C. C. or E. C. C. in staking out a curve on the ground, according to the direction in which the line runs.

for x and y, their values may be laid out simultaneously with corresponding equal chord measurements along the transition curve.

The principle enunciated in the paragraph preceding, enables us to prepare a table of deflection angles according to the following method:

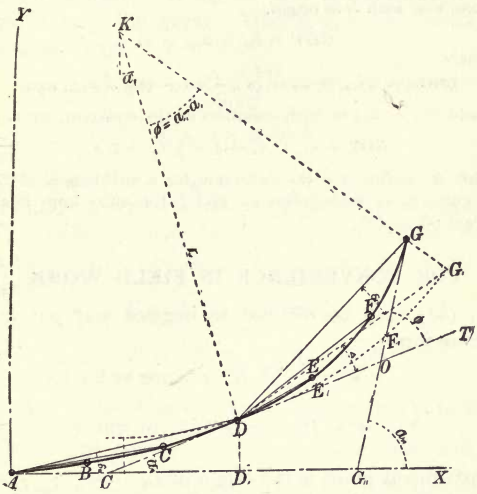

Fig. 3.

Referring to Fig. 3. If the deflection angles from AX to any point be denoted by $\delta\delta_1\delta_{11}$, it will be found by computation that any angle as

$$\text{DAX} = \delta_1 = \frac{\alpha_1}{3} = \frac{\text{L}}{6\text{r}_{11}} \text{ (nearly), *} \qquad (53)$$

in which $r_{11} = DK$ and $L = AD$.

$$\text{ADC}_1 = \text{DCX} - \text{DAX} = \alpha_1 - \frac{\alpha_1}{3} = \frac{\text{L}}{2\text{r}_{11}} - \frac{\text{L}}{6\text{r}_{11}} = \frac{\text{L}}{3\text{r}_{11}} = \qquad (54)$$

---

* Since $\frac{DD_1}{AD_1} = \frac{y_{11}}{x_{11}} = \tan\frac{\alpha_1}{3}$ (nearly).

the angle which CT, a tangent common to the spiral and circular curve with radius $r_{//}$, makes with the chord AD produced. To establish the points E, F and G by deflection at D, from tangent DT we have, from the paragraph already referred to,

$$EDT = \delta + \Delta_{/} \tag{55}$$

and δ = the deflection from AX to B, and $\Delta_{/}$ = the deflection from DT to the point E, for the circular curve with radius $r_{//}$. In the same way with D as origin.

$$GDT = \delta_{//} + \Delta_{//}; = \omega. \tag{56}$$

The angle

$$\mathbf{DGG_{//}} = 2\,\delta_{//} + \Delta_{//} = (\alpha_{//} - \alpha_{/}) - (\delta_{//} + \Delta_{//}), \text{ and} \tag{57}$$

if we add $(\delta_{//} + \Delta_{//})$ to both members of the equation, we have:

$$\mathbf{GOT} = \alpha_{//} - \alpha_{/} = \phi = 3\,\delta_{//} + 2\,\Delta_{//}, \tag{58}$$

in which $\Delta_{//} = L\Delta$, Δ = the deflection for a unit length of DG on the circular curve with radius $r_{//}$, and L the units from D of any point laid off on DG.

FOR CONVENIENCE IN FIELD WORK.

Eq. (53) may be reduced to degrees and put in the following form:

$$\delta^{\circ} = \frac{L}{6r}\,57.3^{\circ}, \text{ whence by Eq. (56)}$$

$$*\quad \omega^{\circ} = D\Delta L \pm \frac{L}{6r}\,57.3^{\circ} \text{ in which} \tag{59}$$

the instrument point is the origin of L.

Δ = deflection angle per foot from tangent for 1° circular curve.

D = degree, or rate of curvature, at position of instrument.

$D_{/}$ = the degree, or rate of curvature, at the point to be located.

ω = deflection from tangent at any point of spiral to locate any other point of spiral (**using the + sign for running toward G, − sign towards A.**)

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* More nearly 57.2956°.

This equation can be still further simplified in application by the following reduction in $\delta^\circ = \frac{L}{6r} 57.3^\circ$, let r = radius due to L measured from the position of the instrument to the point of spiral to be located.

N = the number of chords of equal length (each subtending one degree change in rate of curvature) in the length of L of the spiral. Then if r_0 = the radius of a one-degree curve,

$$r = \frac{r_0}{N} \text{ and } \delta^\circ = \frac{LN\ 9.55^\circ}{5730} = LN \times 0.00166^\circ \quad (60)$$

or

$$\delta' = LN\ 0^\circ 00.1'.$$

If we wish the change in rate of curvature to be any other than one degree to the chord, and denote this change by C, C = the change in degree of curvature, due to one-chord length, and may be either a fraction or whole number; then

$$\delta' = LNC \times 0^\circ 00.1'; \quad (60a)$$

$$\text{or since } C = \frac{D_1}{N},\ \delta' = LD_1 \times 0^\circ 00.1';* \quad (60b)$$

and if one of the chords be fractional, and we indicate the fractional part by $\frac{n}{F}$ in which F is the number of parts into which a chord is divided, and n the number of such parts taken,

$$\delta' = L\left(N + \frac{n}{F}\right) C \times 0^\circ 00.1', \text{ hence} \quad (60c)$$

$$\omega' = \mathrm{L}\left[\mathrm{D}_\Delta \pm \left(\mathrm{N} + \frac{\mathrm{n}}{\mathrm{F}}\right) \mathrm{C} \times \mathrm{o}^\circ \mathrm{oo.1}'\right] \quad (61)$$

is a general formula for the deflection from a tangent, at any point of the transition curve, to locate any other point of the transition curve.

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* See Appendix.

In which case $$C = \frac{D_1 - D}{N + \frac{n}{F}},$$

to be computed and substituted in (61).

If n = 0, $$C = \frac{D_1 - D}{N}$$

and $$\omega' = L\,[D\Delta + (D_1 - D)\ 0°00.1']$$

With instrument at A, $D\Delta = 0$.

To place the line of sight on tangent at any point of the spiral. After backsight on last instrument point, formula (61), for deflection to tangent becomes

$$\omega' = L\left[D\Delta + 2\left(N + \frac{n}{F}\right) C\,0°00.1'\right]$$

when running towards G (Fig. 3), D being the rate of curvature at the last instrument point, and L the length of spiral to backsight. See Eq. (57) $\omega' = L\left[D\Delta - 2\left(N + \frac{n}{F}\right) C\,0°00.1'\right]$ running towards A.

In applying Eq. (61), the more frequent the change points, the more nearly will the resulting curve agree with the theoretical spiral; their distance apart to be not more than 150 to 300 ft. nor "change points" include a central angle of more than 10° to 15°.

A nearer value for the second term in the bracket of (61) is:

$$\left(N + \frac{n}{F}\right) C\,0°\,00'.0998.$$

Example 7. Given, $D = 2$, $\Delta = .3'$, $L = 125$, $N = 2$, $\frac{N}{F} = \frac{1}{2}$, $C = 1$, Chords = 50 ft., to find the deflection from a tangent at C to locate a point $E + 25$ (see Fig. 4). By formulæ (61):

$$\omega' = L\left[D\Delta + \left(N + \frac{n}{F}\right) \times 1 \times 0°00.1'\right]$$

substituting values given above,

$$\omega' = 125\,[2 \times .3' + (2 + \tfrac{1}{2}) \times 1 \times 0°\ 00.1'], \text{ or}$$
$$\omega' = 125\,[.6' + .25'] = 125 \times .85' = 1°\ 46\tfrac{1}{4}'.$$

(See table, Spiral 1.)

The foregoing values for the deflection angles are closely approximate, though the method indicated in connection with Fig. 4 in determining any deflection angle, as δ_n, while less elegant is more nearly exact.

By Eqs. (21) and (26) $\tan \delta_n = \frac{y_n}{x_n}$, (62)

from which we determine δ_n.

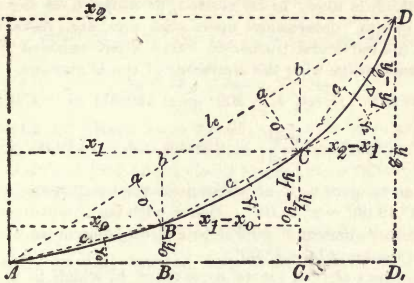

Fig. 4.

"n" having any value, whole number or fractional.

With A as origin we obtain, by (62) the deflection angles to the points B, C, D, E, etc., to any change point, as C, where the degree, or rate of curvature, is D. From C, with backsight on A, deflect $\alpha - \delta_n$ (in which $n = 2$, and α = the central angle of the spiral from A to C) to get on tangent at C, then to locate any point, as E (not shown on Fig. 4), from a tangent at C, deflect

$$* \omega = LD\Delta + \delta_n \qquad (63)$$

δ_n being the deflection angle due to n chords from C, corresponding to the same number of chords from A.

To get on tangent at E:

With backsight on C, deflect $LD\Delta + (\alpha - \delta_n)$, in which $n = 2$, $L = CE$, and α corresponds to the central angle of the spiral from A *for a length* L.

The process is the same as with Eq. (61), though the second factor of the second member is different.

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* It will be seen that the differences between this method and that of Eq. (61), for a curve with $L = 400$ ft. and $\delta = 16°$ is 0°00′03″.

The difference increases with the central angle α. The above method, with change points 200 to 300 ft. apart, is quite accurate and the best for preparing a set of tables — though not so easily applied in field computations as (60) or (61). By assuming "change points" 100 to 250 ft. apart, *the deflection tables* of this book may be extended indefinitely.

The several angles can be computed and tabulated, to any number which is likely to be needed, to conform to any system of "change points" determined upon after $x_0 y_0$, etc., have been computed for the particular transition curve where value of P has been fixed in conformity with the character of the alignment.

EXAMPLE 8. Given $L = 200$; $x_c = 199.91$; $y_c = 4.65$; to locate C from A.

$$\tan \delta_c = \frac{4.65}{199.91} = .023260 \text{ or } \delta_c = 1^\circ\, 19.95'.$$

To get on tangent at C, at which point the total curvature is $4^\circ 00'$: $4^\circ 00' - 1^\circ 19.95' = 2^\circ 40.05'$. Hence with the instrument at C and backsight on A, deflect $2^\circ 40.05'$ to get on tangent at C, where the rate of curvature is $D = 4^\circ 00'$.

From tangent at C to locate some point E, which is, in this case, 200 ft. from C, then

$$\omega = LD\Delta + \delta_n = 200 \times 4 \times .3' + 1^\circ 19.95' = 5^\circ 19.95'$$

and to get on tangent at E, with backsight on C,

$$\omega = LD\Delta + (a - \delta_c) = 200 \times 4 \times .3' + 2^\circ 40.05' = 6^\circ 40.05'.$$

ORDINATES.

§ 10. To determine the ordinates o, o_1, etc. From any chord as l_0, let $aB = o$, $B_1 b = s$, $AB_1 = x_0$, $B_1 B = y_0$, $bAB_1 = \gamma$. Then from Fig. 4, $\frac{s}{x_0} = \tan \gamma$, $s = x_0 \tan \gamma$, $s - y_0 = x_0 \tan \gamma - y_0$,

$\frac{o}{s - y_0} = \cos \gamma$, $o = (s - y_0) \cos \gamma$, or since $s = x_0 \tan \gamma$,

$$o = (x_0 \tan \gamma - y_0 \cos \gamma = x_0 \sin \gamma - y_0 \cos \gamma : \qquad (64)$$

For the distances Aa, etc., $Aa = A_1 B_1$, $\sec \gamma - aB \tan \gamma = x_0 \sec \gamma - o \tan \gamma$. In the same way we may obtain o_1 and Aa_1. To determine the point B, C, etc., by measurement alone: First compute and lay off the distances Aa, Aa_1, etc., then lay off AB and aB simultaneously; next BC and $a_1 C$, etc.; when γ is small the distances bB and $b_1 C$ may be used, distances Ab and Ab_1 being computed and laid off first.

As a check it will be an advantage to compute the length of the long chord as well as the angle it makes with the axis of x, thus:

$$\tan \text{DAD}_1 = \frac{\text{DD}_1}{\text{AD}_1} = \frac{y_n}{x_n}; \qquad (65)$$

in which (n) equals the number of increments or stations between

A and *D*. Let the length of any chord be c_2, c_4, c_n, in which 2, 4, n indicate the 2d, 4th and nth increment.

$$c_2 = (x_2 - x_1) \sec \gamma_2; \quad c_4 = (x_4 - x_3) \sec \gamma_4;$$

$$c_n = (x_n - x_{n-1}) \sec \gamma_n. \tag{66}$$

EXAMPLE 9. Given $x_2 = 209.65$; $x_1 = 120$; $\gamma_2 = 4° 37'$, to find the length of chord, $c_2 = (x_2 - x_1) \sec \gamma_2 = 89.65 \times 1.0032 = 89.93$. For the length of long chord (from origin), $x_2 = 209.65$; $\gamma = 2° 27'$; $x_2 \sec \gamma = 209.65 \times 1.009 = 209.84$.

It is to be observed that the superelevation of the outer rail, in the use of the transition curve, may be made greater or less than that which has been assumed in computing the tables; the only effect it will have is to diminish or increase the assumed value of "i," which is equivalent to increasing or diminishing the velocity, since i and v are inverse functions of each other in the constant P, i.e., it makes the rate of rise of the outer rail to effect superelevation a little greater or less. It is, however, best to introduce the average velocity of the express or fast passenger trains in constructing the tables. Where the location is so constrained that the *EC*'s and *BC*'s of the circular curves are quite close together, it may be necessary to give "i" a smaller value than would be otherwise desirable. A value of 300 or 400 is sufficient for adjustment, and good results may be obtained with a value of 200 when the radius is not greater than 573 feet, since v usually is made to decrease with r.

The beginning and end of the transition curve should be marked by permanent points.

Degree of Curve	Radius	Reciprocal	Degree of Curve	Radius	Reciprocal
D	r	$\frac{1}{r}$	D	r	$\frac{1}{r}$
0°30′	11460.	.00008726	8°00	716.3	.00139626
1°00′	5730.	.00017453	9°00	636.7	.00157079
1°30′	3820.	.00026179	10°00	573.	.00174533
2°00′	2865.	.00034906	11°00	520.9	.00191986
2°30′	2292.	.00043633	12°00	477.5	.00209439
3°00′	1910.	.00052359	13°00	440.8	.00226893
3°30′	1637.1	.00061086	14°00	409.3	.00244346
4°00′	1432.5	.00069808	15°00	382.	.00261799
4°30′	1273.3	.00078540	16°00	358.1	.00279253
5°00′	1146.	.00087267	17°00	337.	.00296706
5°30′	1041.8	.00095993	18°00	318.3	.00314159
6°00′	955.	.00104712	19°00	301.6	.00331613
7°00′	818.8	.00122173	20°00	286.5	.00349066

PROBLEM I.

TO FIND THE SEMI-TANGENTS.

§ 11. Given a circular curve whose radius $= r_{,,}$; the intersection angle $= I$; the semi-tangent $= T$, to unite it with the tangents by means of transition curves whose lengths are $L_{,}$ and L and offsets are $m_{,}$ and m respectively.

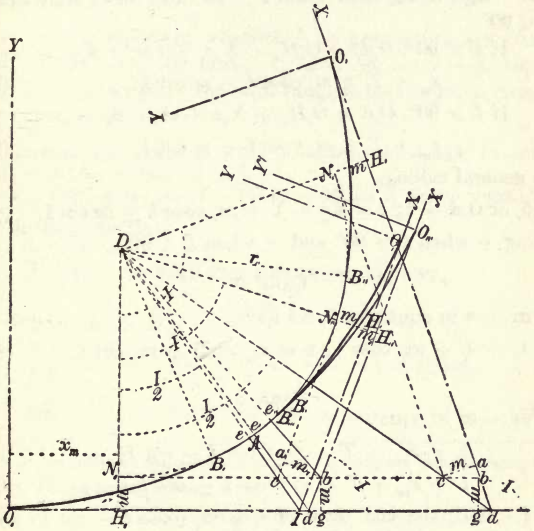

Fig. 5.

CASE 1.

When $m_{,} > m$ (by Fig. 5), if $I < 90°$,

$$O_{,,,}d_{,} = O_{,,,}H_{,,} + N_{,,}c_{,} - a_{,}c_{,} + b_{,}d_{,}$$
$$= x_{m,} + T - m_{,} \cot I + m \operatorname{cosec} I. \qquad (1)$$

If $I > 90°$,

$$O_{,}d = O_{,}H_{,} + N_{,}c + ab + bd =$$
$$x_{m_{,}} + T - m_{,}(-\cot I) + m \operatorname{cose} I; \quad (2)$$
$$O_{,}d = x_{m} + T + m_{,} \cot I + m \operatorname{cose} I; \quad (3)$$

or in general calling,

$$\mathbf{O_{,,,}d,\ or\ O_{,}d = t_{,}\ ;\ t_{,} = x_{m_{,}} + T \mp m_{,} \cot I + m \operatorname{cose} I,} \quad (4)$$

the + sign being used when $I > 90°$ and the − sign when $I < 90°$.

If $I < 90°$, $O_{,}d_{,} = O_{,}H_{,} + N_{,}c_{,} + c_{,}b_{,} - d_{,}g_{,} =$

$$x_{m} + T + m_{,} \operatorname{cose} I - m \cot I; \quad (5)$$

If $I > 90°$, $O_{,}d = O_{,}H_{,} + N_{,}c + cb + dg =$

$$x_{m} + T + m_{,} \operatorname{cose} I + m \cot I. \quad (6)$$

In general calling,

$$\mathbf{O_{,}d,\ or\ O_{,}d = t;\ t = X_{m} + T + m_{,} \operatorname{cose} I \pm m \cot I,} \quad (7)$$

using + when $I > 90°$ and − when $I < 90°$.

CASE 2.

If $\mathbf{m = o}$ in equation 4, we have

$$\mathbf{t_{,} = T + m_{,} \operatorname{cose} I,\ t = x_{m} + T + m_{,} \cot I.} \quad (8)$$

CASE 3.

If $\mathbf{m = m}$ in equation 4,

$$t_{,} = x_{m_{,}} + T + m_{,}(\operatorname{cose} I \pm \cot I), =$$
$$t = x_{m} + T + m(\operatorname{cose} I \pm \cot I). \quad (9)$$

If $I < 90°$, the last term of $t_{,}$ is $m_{,}(\operatorname{cose} I - \cot I)$ and by trigonometry (see Chauvenet's, p. 35).

$$\operatorname{cose} I - \cot I = \frac{1}{\sin I} - \frac{\cos I}{\sin I} = \frac{1 - \cos I}{\sin I} = \tan \tfrac{1}{2} I$$
$$\therefore t_{,} = x_{m_{,}} + T + m_{,} \tan \tfrac{1}{2} I$$

or

$$t_{,} = x_{m_{,}} + (r_{,,} + m_{,}) \tan \tfrac{1}{2} I\ ; \quad (10)$$

similarly, if $I > 90°$,

$$\operatorname{cose} I + \cot I = \frac{1 + \cos I}{\sin I}\ ;$$

but if $I > 90°$,

$$\cos I = -\cos I \text{ and } \cot I = -\cot I$$

$$\operatorname{cosec} I + (-\cot I) = \frac{1 - \cos I}{\sin I},$$

which we have seen $= \tan \frac{1}{2}I$. Hence when $\mathbf{m_{,} = m}$,

$$\mathbf{t_{,} = t = x_m + T + m \tan \tfrac{1}{2}I = x_m + (r_{,,} + m) \tan \tfrac{1}{2}I} \quad (11)^*$$

(See App. I)

is a general equation whether I be greater or less than 90°. If in (11) we make x_m and m each $= o$, then $t = T = r_{,,} \tan \frac{1}{2}I$ = the semi-tangent for a simple circular curve.

EXAMPLE 10. Given $L = 210$; $r = 818.8$; $m_{,} = 2.23$; $I = 40°$; $O_{,}H_{,} = 104.93$; to find t (Fig. 5):
$t = x_{m_{,}} + (r_{,,} + m_{,}) \tan \frac{1}{2}I = 104.93 + 821.03 \times .36397 = 403.75$

With the same data

§ 12. TO FIND THE EXTERNAL SECANT.

CASE 1.

When $\mathbf{m_{,} > m}$,

$$\frac{H_{,}d_{,}}{DH_{,}} = \frac{T + m_{,} \operatorname{cose} I \pm m \cot I}{r_{,,} + m} = \tan H_{,}Dd_{,};$$

$$\mathbf{e_{,}d_{,} = (r_{,,} + m) \sec H_{,}Dd_{,} - r_{,,}.} \quad (12)$$

---

* If we wish to unite two grades by a vertical curve, then, if in Eq. 44, we make $\frac{1}{r_{,}} = o$

and

$$\sin \phi = \sin \tfrac{1}{2}I$$
$$L = 2r_{,,} \sin \tfrac{1}{2}I$$
$$x = L\,(1 -)$$
$$y = \frac{L^2}{6r_{,,}}$$

and Eq. 11, Prob. 1 $\quad t = x_m + (r_{,,} + m) \tan \frac{1}{2}I$

in which $\frac{1}{2}I = \frac{1}{2}$ the angle at which the grades intersect, $r_{,,}$ should have a value of from 5730 to 11460, and represents the minimum radius of curvature at the middle point of the vertical curve, x and y ordinates from the origin or terminus of the vertical curve to its middle point or to any other point.

CASE 2.

If $m = 0$,

$$\frac{N_{,}b_{,}}{r_{,,}} = \frac{T + m_{,} \operatorname{cose} I}{r_{,,}} = \tan N_{,}Db_{,} ;$$

$$e_{,,}b_{,} = r_{,,} \sec N_{,}Db_{,} - r_{,,}. \qquad (13)$$

CASE 3.

$$m_{,} = m, e_{,}d_{,} = (r_{,,} + m) \sec \tfrac{1}{2}I - r_{,,} \qquad (14)$$

If, in Eq. 14, we make $m = o$, we have for the external secant of a simple circular curve:

$$ed = r_{,,} (\sec \tfrac{1}{2}I - 1) = ec.$$

If we wish the transition curves $B_{,,,}O_{,,,}$ and $B_{,}O_{,,}$ to become a tangent to each other at some point A with a minimum radius of curvature $= r_{,,}$ = the radius of the circular portion $B_{,,,}B_{,,}$ with $B_{,,,}B_{,}$ elided, then $B_{,,,}$ is coincident with $B_{,}$.

$$B_{,}DN_{,} = \alpha = I - \frac{L_{,}}{2r_{,,}}, B_{,,,}DN_{,,} = \alpha_{,} = L - \frac{L}{2r_{,,}}, \qquad (15)$$

$$\alpha_{,} + \alpha = I \qquad (16)$$

$$O_{,,,}B_{,,,} = L_{,} = 2r_{,,} (I - \alpha) = \sqrt{24r_{,,}m_{,}} \qquad (17)$$

$$O_{,}B_{,} = L = 2r_{,,} (I - \alpha_{,}) = \sqrt{24r_{,,}m} \qquad (18)$$

$$N_{,,}H_{,,} = m_{,} = \frac{L_{,}^{2}}{24r_{,,}} \qquad (19)$$

$$N_{,}H_{,} = m = \frac{L^{2}}{24\, r_{,,}} \qquad (20)$$

If $m_{,} = m$ then $L_{,} = L = 2r_{,,} \tfrac{1}{2}I$, the point A is at the middle point of the curve and $L_{,} = \tfrac{1}{2}$ the total length of the curve from *B.S.*, through *C. C.* to *E. S.*

Since $\alpha_{,}$ and α are expressed in arc in the above formulas, while the intersection angle is measured in degrees and minutes, the latter will have to be reduced to arc at

radius = 1 before entering the formulas, and after calculations are made the result expressed again in degrees and minutes to apply formulas for semi-tangents, external secants, etc.

PROBLEM II.

TO FIND THE LOCATION OF THE OFFSET "f".

§ 13. Given two curves with radii $r_{,}$ and $r_{,,}$, a distance $D_{,}D_{,,} = d$ joining the points $D_{,}$ and $D_{,,}$ also the angles $BD_{,}D_{,,} = \rho$ and $CD_{,,}D_{,} = \theta$, to find the points $A_{,}$ and $A_{,,}$ at which a line drawn through the centers $r_{,}$ and $r_{,,}$ at B and C will cut the curves $A_{,}D_{,}$ and $A_{,,}D_{,,}$.

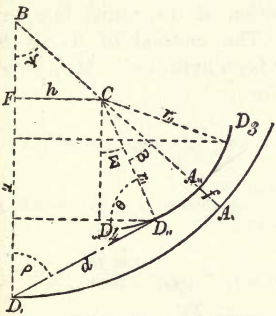

Fig. 6.

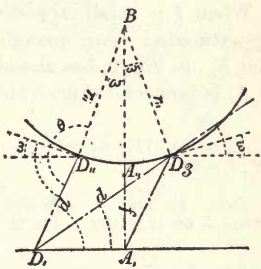

Fig. 7.

From Fig. 6 we have

$$FC = h = d \sin \rho - r_{,,} \sin (180 - (\rho + \theta)), \quad (1)$$

$$BF = r_{,} - [d \cos \rho + r_{,,} \cos (180 - (\rho + \theta))]$$

$$\frac{FC}{BF} = \tan \Psi = \frac{d \sin \rho - r_{,,} \sin (180 - (\rho + \theta))}{r_{,} - [d \cos \rho + r_{,,} \cos (180 - (\rho + \theta))]} \quad (2)$$

and since $BC = r_{,} - (r_{,,} + f)$,

$$\frac{FC}{BC} = \frac{h}{r_{\prime} - (r_{\prime\prime} + f)} = \frac{d \sin \rho - r_{\prime\prime} \sin (180 - (\rho + \theta))}{r_{\prime} - (r_{\prime\prime} + f)} = \sin \Psi \quad (3)$$

$$D_{\prime\prime}CD_4 = \Sigma = 180 - (\rho + \theta), \Psi - \Sigma = \omega = D_{\prime\prime}CA_{\prime\prime}; \quad (4)$$

whence $$r_{\prime\prime}\omega = A_{\prime\prime}D_{\prime\prime},$$

in which ω is the arc of a circle with radius = 1. (5)

If $\Sigma > \Psi$, then the point $A_{\prime\prime}$ lies between $D_{\prime\prime}$ and D_3 and the distance $A_{\prime\prime}D_3$ is measured from D_3 towards $D_{\prime\prime}$ to $A_{\prime\prime}$.

If $\Psi > \Sigma$, the point $A_{\prime\prime}$ lies beyond $D_{\prime\prime}$ and the distance $A_{\prime\prime}D_{\prime\prime}$ is measured from $D_{\prime\prime}$ to $A_{\prime\prime}$ whence $A_{\prime\prime}$ is established. On a perpendicular to a tangent at $A_{\prime\prime}$ lay off f, and establish $A_{\prime}$.

When f is small, the direction of the radial line can be estimated near enough. The method of fixing $B_{\prime}$ and $B_{\prime\prime}$, in Fig. 2, has already been indicated. If $r_{\prime} = \infty$, $D_{\prime}A_{\prime}$ is tangent. Fig. 7 applies.

EXAMPLE 11. $r_{\prime\prime} = 1000$; $r_{\prime} = 600$; $d = 300$; $\rho = 70°$; $\theta_{\prime} = 96°$; $f = 50_{\prime}$:

First, to find $h = d \sin \rho - r_{\prime\prime} \sin (180 - (\rho + \theta_{\prime})) = 300 \times .93969 - 60 \times .24192 = 136.75 = FC$.

Second, to find $\sin \Psi = \dfrac{h}{r_{\prime} - (r_{\prime\prime} + f)} = \dfrac{136.75}{1000 - (600 + 50)}$

$$= \frac{136.75}{350} = 3097; \frac{FC}{BC} = \sin 23°00.$$

Third, to find $180 - [70° + 96°] = 14° = \Sigma$; $\Psi - \Sigma = \omega = 23° - 14° = 9°$ ω reduced to arc $= \dfrac{9}{57.29} = .1571$.

$r\omega = 600 \times .1571 = 94.26 = A_{\prime\prime}D_{\prime\prime}$, and since $\Psi > \Sigma$, $A_{\prime\prime}D_{\prime\prime}$ is measured from $D_{\prime\prime}$ towards D_3.

If $\Sigma = D_4CD_3$, then $\Sigma > \Psi$ and $A_{\prime\prime}D_3$ would be measured from D_3 towards $D_{\prime\prime}$ to establish point $A_{\prime\prime}$; $A_{\prime}$ is on a perpendicular to a tangent to the curve with radius $r_{\prime\prime}$ at $A_{\prime\prime}$.

PROBLEM III.

COMPOUND CURVES.

§ 14. Given two circular curves with radii r_1 and r_3, respectively, whose centers are apart a distance $AC = b = r_1 - (r_3 + f_{II})$, and which are separated from each other a distance $D_1D_{II} = f_{II}$. It is desired to introduce between them a third curve with radius r_{II}, less than r_1 and greater than r_3 and to join the curve with radius r_{II} with those having radii r_1 and r_3 by means of transition curves:

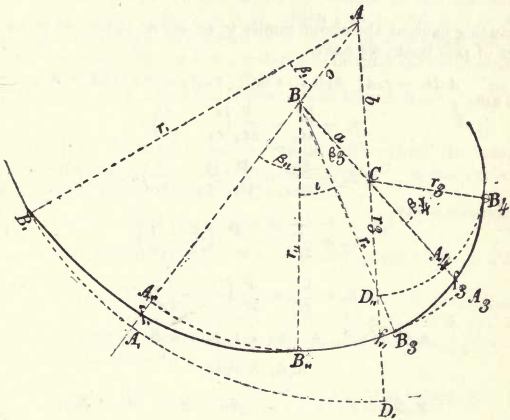

Fig. 8.

By the figure $AD_1 = AA_1 = AB_1 = r_1 = AC + CD_{II} + D_{II}D_1 = b + r_3 + f_{II}$; whence

$$AC = b = r_1 - (r_3 + f_{II}); \tag{1}$$

$$AB = c = r_1 - (r_{II} + f_1); \tag{2}$$

$$BC = a = r_{II} - (r_3 + f_3); \tag{3}$$

a, b and c form the sides of a triangle ABC in which

$$f_1 = \frac{L_1^2}{24}\left(\frac{1}{r_{II}} - \frac{1}{r_1}\right); \; f_3 = \frac{L_3^2}{24}\left(\frac{1}{r_3} - \frac{1}{r_{II}}\right); \tag{4}$$

$$L_{\prime} = P\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right);\ L_3 = P\left(\frac{1}{r_3} - \frac{1}{r_{\prime\prime}}\right). \tag{5}$$

Any angle A may be found by the formulæ,

$$\text{Ver } A = 2\,\frac{(s-b)\,(s-c)}{bc} \quad \text{in which } s = \tfrac{1}{2}(a+b+c); \tag{6}$$

$$\text{Sin } B = \frac{b}{a}\,\text{sin } A; \tag{7}$$

$$\text{Sin } C = \frac{c}{a}\,\text{sin } A. \tag{8}$$

Reducing each of the above angles to an arc as indicated in another part of this book, we have

$$A_{\prime}D_{\prime} = r_{\prime}A;\ A_4D_{\prime\prime} = r_3C;\ A_{\prime\prime}A_3 = r_{\prime\prime}\,(180 - B); \tag{9}$$

the arc

$$\beta_4 = \frac{L_3}{2r_3} = \frac{P}{2r_3}\left(\frac{1}{r_3} - \frac{1}{r_{\prime\prime}}\right); \tag{10}$$

$$\beta_3 = \frac{L_3}{2r_{\prime\prime}} = \frac{P}{2r_{\prime\prime}}\left(\frac{1}{r_3} - \frac{1}{r_{\prime\prime}}\right); \tag{11}$$

in the same way,

$$\beta_{\prime\prime} = \frac{L_{\prime}}{2r_{\prime\prime}} = \frac{P}{2r_{\prime\prime}}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right); \tag{12}$$

$$* \beta_{\prime} = \frac{L_{\prime}}{2r_{\prime}} = \frac{P}{2r_{\prime}}\left(\frac{1}{r_{\prime\prime}} - \frac{1}{r_{\prime}}\right); \tag{13}$$

$$A_{\prime}B_{\prime} = r_{\prime}\beta_{\prime};\ A_{\prime\prime}B_{\prime\prime} = r_{\prime\prime}\beta_{\prime\prime};\ A_3B_3 = r_{\prime\prime}\beta_3;$$

$$A_4B_4 = r_3\beta_4; \tag{14}$$

$$B_{\prime\prime}B_3 = r_{\prime\prime}\,\iota;\ \dagger\ \iota = (180 - B) - (\beta_{\prime\prime} + \beta_3); \tag{15}$$

$$B_{\prime}D_{\prime} = r_{\prime}A + r_{\prime}\beta_{\prime} = r_{\prime}\,(A + \beta_{\prime}); \tag{16}$$

$$D_{\prime\prime}B_4 = r_3\,(C + \beta_4). \tag{17}$$

Example 12. Given $r_{\prime} = 2865$; $r_{\prime\prime} = 1146$; $r_3 = 716.3$; $f_{\prime\prime} = 20$; $P = 171900$; $L_{\prime} = 90$; $L_3 = 90$ $\therefore f_{\prime} = f_3 = .17$ to join the circular curves B_1D_1 and $B_4D_{\prime\prime}$ by Prob. III. First finding a, b and c by (1 — 3) we have by (6) ver $A = 3°\,44'$; arc $A = .6516$; by (7) $(180 - B) = 18°\,48'$; arc $(180 - B) = .32811$; by (8) $C = 15°\,06'$;

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* For determining large central angles of $\beta_{\prime}$, $\beta_{\prime\prime}$, β_3 and β the method of § 4 is to be preferred.

† See Appendix I.

arc $C = .26054$; $A_4D_\prime = 2865 \times .06516 = 186.68$; $A_4D_{\prime\prime} = 716.3 \times .26054 = 186.77$; $A_{\prime\prime}A_3 = 1146 \times .3287 = 376.69$; $\beta_4 = \frac{90}{2(716.3)} = .6282$; $\beta_3 = \beta_2 = \frac{90}{2(1146)} = .03926$; $\beta_\prime = \frac{90}{2(2865)} = .0157$; $A_\prime B_\prime = A_{\prime\prime}B_{\prime\prime} = A_3B_3 = A_4B_4 = 45.0$; $\iota = 18° 48' - (2° 15' + 2° 15') = 14° 18'$ arc $\iota = .25017$; $B_{\prime\prime}B_3 = 1146 \times .25017 = 286.7$; $B_\prime D_\prime = 2865\ (.06516 + .0157) = 231.67$; $B_4D_{\prime\prime} = 716.3\ (.26054 + .06282) = 231.62$.

If we make $f_\prime = 0$ and $r_\prime = r_{\prime\prime}$, then $c = 0$, $C = 0$, $f_{\prime\prime}$ and $\beta_{\prime\prime} = 0$, $b = a$ and is coincident with it, $A = 180 - B$ and the problem reduces to uniting $A_{\prime\prime}A_3$ with $D_{\prime\prime}B_4$ by means of the transition curve. If $r_{\prime\prime} = r_\prime$, $c = 0$, $C = 0$, in which case $f_\prime$ and $f_3 = 0$; a is equal to and coincides with b and $A = 180 - B = 0$ equations for vers and sines $= 0$. The problem reduces to uniting curves $B_\prime D_\prime$ and $B_4D_{\prime\prime}$ by means of a transition curve whose length $L_2 = \sqrt{\frac{24 f_{\prime\prime}}{\frac{1}{r_\prime} - \frac{1}{r_3}}}$

If f_3 and $f_\prime = 0$, while $r_\prime$, $r_{\prime\prime}$ and r_3 retain their values, the transition curves disappear and the curve with radius $r_{\prime\prime}$ compounds at A_4 and $A_\prime$ with the curves having radii $r_\prime$ and r_3.

$$180 - B \text{ is the central angle of } A_{\prime\prime}A_3; \tag{18}$$

$$C = \text{that of } D_{\prime\prime}A_4; \qquad A = \text{that of } A_\prime D_\prime. \tag{19}$$

If the compound circular curve, Prob. III, be of two centers with radii $r_{\prime\prime}$ and $r_\prime$, then Eq. 12 and 13 give the value of $\beta_\prime$ and $\beta_{\prime\prime}$ if the central angles be not exceeding 10 or 15 degrees and the radii do not exceed 955 feet; otherwise

Let $a_{\prime\prime} - a_\prime = \beta_{\prime\prime} + \beta_\prime = I$ and $r_{\prime\prime}$ and $r_\prime$ be known; then, Eq. (6) page 5

$$P = \frac{2\,(a_{\prime\prime} - a_\prime)}{\frac{1}{r_{\prime\prime}^2} - \frac{1}{r_\prime^2}} \tag{20}$$

and by (6)

$$a_{\prime\prime} = \frac{P}{2r_{\prime\prime}^2},$$

$$a_\prime = \frac{P}{2r_\prime^2},$$

and f computed by Eq. (37) then by § 4

$\beta_{\prime\prime}$ and $\beta_\prime$ can be obtained, whence $r_{\prime\prime}\beta_{\prime\prime}$ = the length of one branch of the curve and $r_\prime\beta_\prime$ the other. Then to find the *B. C.*, *C. C.* and *E. C.* of the compound circular curve, having

$$a_{\prime\prime} - a_\prime = \beta_{\prime\prime} + \beta_\prime = I. \tag{21}$$

$r_{//} \tan \frac{1}{2}\beta_{//}$ = the semi-tangent for $r_{//}\beta_{//}$ and $r_{/} \tan \frac{1}{2}\beta_{/}$ = the semi-tangent for $r_{/}\beta_{/}$

whence $T_{//} + T = r_{//} \tan \frac{1}{2}\beta_{//} + r_{/} \tan \frac{1}{2} B_{/}$ = the hypotenuse of a triangle in which one side and three angles are known, viz.: $T_{//} + T_{/}$; $\beta_{//}$ and $\beta_{/}$ and $I = \beta_{//} + \beta_{/}$ and for any side opposite $\beta_{//}$ we have

$$b_{/} = (T_{//} + T) \sin \beta_{//}.$$

By a similar process we may find $b_{//}$ whence $T_{//} + b_{//}$ = the semi-tangent adjacent $\beta_{//}$ and $T_{//} + b_{/}$ = the semi-tangent adjacent $\beta_{/}$ which fixes the *B. C.* and *E. C.* and $r_{/}\beta_{/}$ the distance on the curve to *C. C.* The distance $r_{/}\beta_{/}$ assumed to be measured on the curve.*

If it be desired to introduce transition curve at the extremities of this compound curve, the formulas of Problem I are applicable by writing

$$T_{//} + b_{//} \text{ and } T + b_{/} \text{ for } T.$$

When the rate of curvature of the branches of compound curve differ from each other by less than two degrees per station of 100 feet the transition curves may be omitted at the compound points and introduced only at the ends where the curve merges into the tangents.

The semi-tangent to a compound curve of more than two centers may be computed by latitudes and departures.

Case 2.

If it be desired to introduce a third circular curve, exterior to and joining the other two fixed circular curves, by means of spiral arcs, all turning in the same direction, then the conditions are indicated in Fig. (8a) and the solution similar to the first case. P has fixed value from which $f_{/}$, f_3, $L_{/} = B_{/}B_{//}$ and $L_3 = B_3B_4$ are computed. If, however, we wish to displace the third circular curve by spiral arcs, tangent each other at some point O, having a maximum radius of curvature $= r_{//}$, then the first approximate solution will be obtained as follows: By Fig. (8a) in the triangle ABC let the side $AB = r_{//} - r_{/}$, $BC = r_{//} - r_3$ and $AC = r_{/} + r_3 \pm f$, the $\pm$ sign depending on whether AC is greater or less than $r_{/} + r_3$. Solving for angle B, we have, approximately, the central angle of arc $A_{//}A_3$ and $r_{//}B$ = arc $A_{//}A_3$. Let $L_{0/}$ and L_{03} = the

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* The chord of a 10° curve with radius = 573 ft. = 49.99 ft. for an arc of 50 feet.

length of the spiral arcs computed from this data. Since $\beta_{//} : \beta_3 = r_3 : r_{/}$ then $\frac{L_{0/}}{2} = r_{//}\beta_{//} = A_{//}O_{/}$, $\frac{L_{03}}{2} = r_{//}\beta_3 = A_3O_{//}$.

$A_3O_{//} + A_{//}O_{/} < A_{//}A_3$, generally, by an amount $O_{/}O_{//}$. Make $O_{/}O = \frac{O_{/}O_{//}r_3}{r_{/} + r_3}$ and $O_{//}O = O_{/}O_{//} - O_{/}O$ then $2\left(\frac{L_{0/}}{2} + O_{/}O\right) = L_{/}$

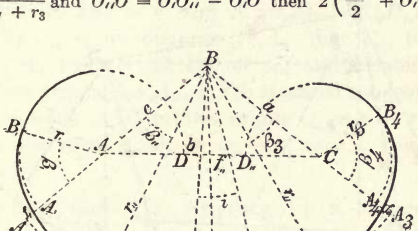


Fig. 8a.

$2\left(\frac{L_{03}}{2} + O_{//}O_{/}\right) = L_3$ nearly, compute $f_{/}$ and f_3 and make the sides of the triangle, AB and AC, respectively $= r_{//} - (r_{/} + f_{/})$ and $r_{//} - (r_3 + f_3)$ and compute $OA_{//}$ and OA_3 repeatedly till $= xf_{/}$ and xf_3, or $O_{/}O_{//} = 0$ (or iota $= 0$). If we make $r_{//}$ infinitely great, then $A_{//}A_3$ will be a straight line and O the common origin of the spirals in which $r_{/} + r_3 \pm f_{//} = xm_{/} + xm_3$ and we would have a special value of P for these particular curves. For a compound circular curve $f_{/}$ and $f_3 = 0$ as in the first part of this case.

PROBLEM IV.

§ 15. Given two curves turning in the same direction, whose centers are a distance apart $= D_{/}D_{//}$ and whose radii are $r_{/}$ and $r_{//}$. To fix the position of a tangent $A_{/}A_{//}$ and connect it with the circular curves $A_{//}C_{/}$ and $A_4C_{//}$ by means of transition curves having a fixed value of P (Fig. 9). Let $C_{/}C_{//}$ be a line joining any two points $C_{/}$ and $C_{//}$ of the circular curves with radii $r_{/}$

and $r_{,,}$; measure the angles $D_{,}C_{,}C_{,,}$, $C_{,}C_{,,}D_{,,}$ and line $C_{,}C_{,,}$. By traverse we find the angles $C_{,}D_{,}D_{,,}$, $C_{,,}D_{,,}D_{,}$ and distance $D_{,}D_{,,}$ between the centers of the curves $r_{,}$

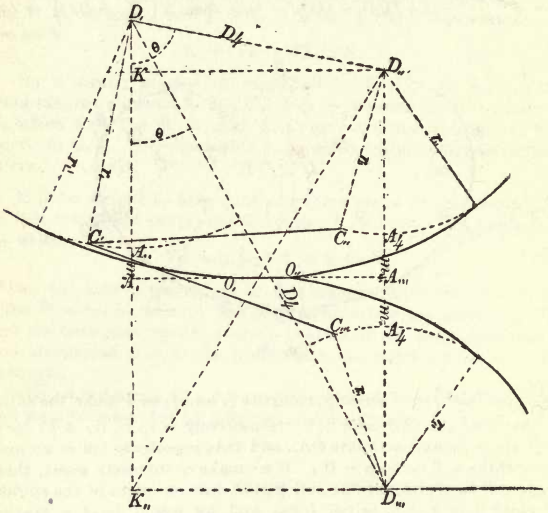


Fig. 9.

and $r_{,,}$. If from $D_{,}$ and $D_{,,}$ we let fall perpendiculars to an imaginary tangent passing through the origins of the transition curves, by the conditions of the problem the length of these perpendiculars will be:

$$D_{,}A_{,} = r_{,} + m_{,} \text{ and } D_{,,}A_{,,,} = r_{,,} + m_{,,},$$

and any distance $D_{,}K_{,} = (r_{,} + m_{,}) - (r_{,,} + m_{,,})$, (1)

and if we denote the distance $D_{,}D_{,,}$ by D_4, then

$$\frac{D_{,}K_{,}}{D_{,}D_{,,}} = \cos\theta = \frac{(r_{,} + m_{,}) - (r_{,,} + m_{,,})}{D_4} \qquad (2)$$

$C_{\prime}D_{\prime}A_{\prime\prime}$ = difference of $C_{\prime}D_{\prime}D_{\prime\prime}$ and θ. $C_{\prime}D_{\prime}A_{\prime\prime}$ reduced to arc and multiplied by $r_{\prime}$ = the distance $C_{\prime}A_{\prime\prime}$ to establish $A_{\prime\prime}$; from $A_{\prime\prime}$ lay off $m_{\prime}$ normal to the circle and establish $A_{\prime}$. In the same way $C_{\prime\prime}D_{\prime\prime}A_4$ = the difference between $180° - \theta$ and $C_{\prime\prime}D_{\prime\prime}D_{\prime}$. $C_{\prime\prime}D_{\prime\prime}A_4$ multiplied by $r_{\prime\prime}$ = arc distance $C_{\prime\prime}A_4$ from $C_{\prime\prime}$ to A_4 to establish A_4, from which lay off $m_{\prime\prime}$ and establish $A_{\prime\prime\prime}$; then a line through $A_{\prime}A_{\prime\prime\prime}$ is the required tangent.

The distance $A_{\prime}O_{\prime}$ to the origin $O_{\prime} = x_{\prime} - r_{\prime} \sin \alpha_{\prime}$ and the distance $A_{\prime\prime\prime}O_{\prime\prime} = x_{\prime\prime} - r_{\prime\prime} \sin \alpha_{\prime\prime}$; whence the distance

$$O_{\prime}O_{\prime\prime} = D_{\prime}D_{\prime\prime} \sin \theta - [(x_{\prime\prime} - r_{\prime\prime} \sin \alpha_{\prime\prime}) + (x_{\prime} - r_{\prime} \sin \alpha_{\prime})]. \quad (3)$$

If we wish the distance $O_{\prime}O_{\prime\prime}$ = zero, make

$$D_{\prime}D_{\prime\prime} = [(x_{\prime\prime} - r_{\prime\prime} \sin \alpha_{\prime\prime}) + (x_{\prime} - r_{\prime} \sin \alpha_{\prime})] \operatorname{cose} \theta. \quad (4)$$

This gives the shortest distance possible between the centers of the circular curves when transition curves are introduced. The transition curves may have different values of P provided $A_{\prime}O_{\prime} + A_{\prime\prime\prime}O_{\prime\prime}$ = or < than $A_{\prime}A_{\prime\prime\prime}$. **If the curves are reversed,** $D_{\prime}K_{\prime\prime} = (r_{\prime} + m_{\prime}) + (r_{\prime\prime} + m_{\prime\prime})$ and we find the value of θ by completing the traverse $C_{\prime}C_{\prime\prime\prime}D_{\prime\prime\prime}D_{\prime}C_{\prime}$.

EXAMPLE 13. Given $D_{\prime}C_{\prime}C_{\prime\prime} = 85°$; $C_{\prime}C_{\prime\prime}D_{\prime} = 100$; $C_{\prime}C_{\prime\prime} = 300$: $r_{\prime} = 1146$; $r_{\prime\prime} = 573$. First, to find $D_{\prime}D_{\prime\prime} = D_4$. By traverse with $D_{\prime}$ as initial point.

From	Course	Distance	Lat. $N+$	Lat. $S-$	Dep $E+$	Dep $W-$	To
$D_{\prime}$	South	1146.0	.....	1146.0	.....	.....	$C_{\prime}$
$C_{\prime}$	$N85° E$	300.0	26.1	.....	298.9	.....	$C_{\prime\prime}$
$C_{\prime\prime}$	$N\ 5° E$	573.0	570.8	.....	49.9	.....	$D_{\prime\prime}$

$$+ 596.9 \quad - 1146.0 \quad + 348.8$$
$$596.9$$
$$- 549.1 \quad + 348.8$$

$$D_iD_{ii} = D_4 = \sqrt{(549.1)^2 + (348.8)^2} = 650.5$$

$m_i = .13$ $m_{ii} = 1.05$ $r_i + m_i = 1146.13$ $r_{ii} + m_{ii} = 574.05$

$$\cos\theta = \frac{(r_i + m_i) - (r_{ii} + m_{ii})}{D_4} = \frac{572.08}{650.5} = 0.87944 = \cos 28° 25'$$

$$\tan C_iD_iD_{ii} = \frac{\text{Dif. dep.}}{\text{Dif. lat.}} = \frac{348.8}{549.1} = 635.22 = \tan 32° 25'$$

$32° 25' - 28° 25' = 4°$ $\therefore$ $D_iA_i = r_i + m_i$ bears $S\ 4°\ E.$

$A_{ii}A_i = m_i = .13'.$ By the table $C_{ii}D_{ii}$ bears $N\ 5°\ E.$

$D_{ii}A_{iii} = r_{ii} + m_{ii}$; parallel to D_iA_i; bears $S\ 4°\ E.$ $C_{ii}D_{ii}A_{iii} = 9°$

$$A_4A_{iii} = 1.05 \quad A_iA_{iii} = D_{ii}K_i = D_4 \sin\theta$$

$$= A_4A_{iii} = 650.5 \times .4759 = 309.5$$

$$O_iO_{ii} = D_iD_i \sin\theta - [(x_{ii} - r_{ii} \sin a_{ii}) + (x_i - r_i \sin a)].$$

Let $x_{ii} = 119.90$; $a_{ii} = 6°$ $x_i = 60'$; $a_i = 1° 30'$

$$O_iO_{ii} = 309.5 - [59.95 + 30] = 309.5 - 89.95 = 219.55$$

to make $O_iO_{ii} = 0$

$$\tan\theta = \frac{89.95}{572.08} = .15273; = 8° 56'$$

$A_iA_{iii} = 89.95$ cose θ, or $D_iD_{iii} = 89.95 \times 6.4398 = 579.25.$

PROBLEM V.

OLD TRACK.

§ 16. To introduce the transition curve in alignment where circular curves have been run.

In Fig. 10, suppose ABC, a simple circular curve, to have been run tangent to the line OG_{ii} at A with radius $r_i = D_iA = D_iB$; it is desired to introduce a spiral whose greatest curvature has a radius $r_{ii} = D_{ii}A_{ii} = D_{ii}B_{ii} = D_{ii}B < r_i$. From the tables or by computation we have m depending on the value of r_{ii} and L_{ii}; with given values of m, r_{ii} and r_i we have from the figure

$$AD_i = A_iA_{ii} + A_{ii}D_{ii} + D_{ii}D_i \cos D_{ii}D_iK;$$

or

$$r_i = m_i + r_{ii} + (r_i - r_{ii}) \cos\phi \therefore$$

$$m = r_i - r_{ii} - (r_i - r_{ii}) \cos\phi$$

$$m = r_i - r_{ii}\,(1 - \cos\phi) = (r_i - r_{ii}) \text{ ver } \phi$$

$$\text{ver } \phi = \frac{m}{r_i - r_{ii}}; \quad r_i\ \phi = AB; \quad r_{ii}\ \phi = A_{ii}B$$

AB = the distance to measure from A = PC to locate B; $BA_{//}$ the distance to measure from B on $BA_{//}$ to locate $A_{//}$. From m and $r_{//}$ we have

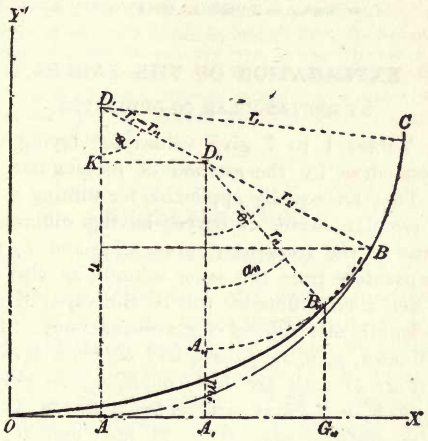


Fig. 10.

$$2r_{//}\alpha_{//} = L = \sqrt{24mr_{//}}, \text{ squaring, } 4r_{//}^2\alpha_{//}^2 = 24mr_{//}$$

or
$$\alpha_{//}^2 = \frac{6m}{r_{//}}\,;\ \alpha_{//} = \sqrt{\frac{6m}{r_{//}}}\,;\ r_{//}\alpha_{//} = A_{//}B_{//} = \frac{L}{2}$$

$$r_{//}\,(\phi_{//} - \alpha_{//}) = BB_{//};\ (r_{/} - r_{//})\sin\phi = A_{/}A.$$

The rate of curvature of $r_{//}$ should not be more than from 1° to 2° greater than that of $r_{/}$ (when possible) for curves of a curvature less than 10°; 2° to 3° difference for 10° to 15° rate of curvature; 3° to 5° difference for 15° to 20° rate of curvature.

EXAMPLE 14. Given $r_{/} = 1146$, $r_{//} = 955$, $m = 1.52$, to find ϕ; $\alpha_{//}$, AB, $BB_{//}$ and L.

$$\text{ver}\,\phi = \frac{m}{r_{/} - r_{//}} = \frac{142}{1146 - 955} = 0.00743 = \text{ver } 6°\,59_{/}\,;\ \text{arc } \phi = .12188$$

$r_{,}\phi = .1219 \times 1146 = AB$, $r_{,,}\phi = 1219 \times 955 = 115.8 = A_{,,}B$

$$\text{arc } \alpha_{,,} = \sqrt{\frac{6m}{r_{,,}}} = \sqrt{\frac{1.42 \times 6}{955}} = .0944 \; \alpha_{,,} = 5° \; 24'$$

$r_{,,}(\phi - \alpha) = 955\ (.1216 - .0944) = 25.7 = B_{,,}B$

$L = 2r_{,,}\alpha_{,,} = 2\ (955 \times .0944) = 180.2$

EXPLANATION OF THE TABLES.

BY RECTANGULAR CO-ORDINATES.

§ 17. Tables 1 to 7 give values for laying out the transition curve by the method of rectangular co-ordinates. They are equally applicable for uniting a tangent with a circular curve, or curves having different radii, by means of the transition curve. $L_{,,}$ and $L_{,}$ may be taken separately from the same column, as also may $\alpha_{,,}$ and $\alpha_{,}$, and their difference will be the value of L and ϕ for the length and central angle respectively. The several ordinates, x, y, x_f, y_f, are laid off from $B_{,}$ or $B_{,,}$ as origin (Fig. 2) with arc $B_{,}A_{,}$ or $B_{,,}A_{,,}$ as axis, — the same as if $B_{,}$ or $B_{,,}$ were written for A in Fig. 3, and the successive stations were B, C, D, etc., and $B_{,}$, $C_{,}$, $D_{,}$, etc., successive points x, $x_{,}$, $x_{,,,}$, etc., with corresponding y, $y_{,}$, $y_{,,,}$ etc., values normal to the curved axis $A_{,}B_{,}$ $A_{,,}B_{,,}$ in the same manner as if $A_{,}B_{,}$ were tangent. P and v are taken of such values as to avoid introducing fractions in L. L is supposed to be measured on the curve, but since the chords are generally quite short, the sum of their lengths is but little less than that of the curve, hence no allowance is made for the length of the curve being in excess of the sum of the lengths of the chords.

Example 1. Given $L = 120$; $r_{,} = 1432.5$; $r_{,,} = 716.3$ (Table II), then $\alpha_{,,} - \alpha_{,} = 9° \; 36' - 2° \; 24' = 7° \; 12' = \phi$ with E as origin. At the end of the first chord length from E towards I we have $x = 30'$, $y = .03$ = co-ordinates for F; $x_{,} = 60$, $y_{,} = .21$ = co-ordinates for G; $x_{,,} = 90$, $y_{,,} = .70$ = co-ordinates for H; $x_3 = 119.98$, $y_3 = 1.67$ = co-ordinates for I; $x_f = 60$; $y_f = .21$; $f = .42$.

The method of laying out these co-ordinates is shown in Fig. 4, in which the origin A corresponds to the point E in this example, and $AD_{/}$ becomes a curved axis with a radius of 1432.5.

If the transition curve were laid off from $B_{//}A_{//}$, Fig. 2 as axis and $B_{//}$ as origin, the above values x, etc., and y, etc., would be just the same except they would be laid off from the convex side of $A_{//}B_{//}$ instead of from the concave side, as was the case with $A_{/}B_{/}$ as axis. If the curvature of the circular curve is of fractional degree the value of f and the last values of x, y and a will have to be computed by the formulæ at the head of the respective columns in Spiral Table I.

BY DEFLECTION.

(Or Polar Co-ordinates.)

§ 18. Given a tangent at any point of a transition curve as D to locate any other points as A, B, C, E, F, G and H. As in the case of the Tables for rectangular co-ordinates they are equally applicable for locating the points of the transition curve uniting a tangent to the circular curve or circular curves of different radii with each other. (Fig. 3.)

EXAMPLE 2. Let $L_{//} - L_{/} = L = 120$ be the length of the transition curve; $r_{/} = 1910$; $r_{//} = 716.3$ the radii of the circular curves to be united by L by deflections from a tangent at D, where the curvature corresponds to $r_{/}$. The tangent will be common to the circle and transition curve whose rate of curvature "D" (Table II) = 3°. DE, EF, FG, etc., being chords of the transition curve each = 30 feet. The tangent at D is a tangent to the circular arc with $r_{/} = 1910$. Then by the formula $\omega = L[D\Delta + N \times 0° 00.1']$ in which, if we write $L = 30$, 60, 90 and 120; and $N = 1$, 2, 3 and 4 successively, $D = 3$, $\Delta = 0.3'$, then the deflection from tangent at

D to locate E is $\omega = 30\,[3 \times .3' + 1 \times 0° 00.1'] = 0° 30'$

D to locate F is $\omega = 60\,[3 \times .3' + 2 \times 0° 00.1'] = 1° 06'$

D to locate G is $\omega = 90\,[3 \times .3' + 3 \times 0° 00.1'] = 1° 48'$

D to locate H is $\omega = 120\,[3 \times .3' + 4 \times 0° 00.1'] = 2° 36'$

with measurement from D to E thence E to F, etc. If we wish to locate the points C, B and A from a tangent at D, then the deflection for any point C, for example, = the deflection for 30 feet for a 3-degree curve minus the deflection for the transition curve from A to B by formula (61).

$$\omega = L[D\Delta - N \times 0^\circ\ 00.1'], \text{ whence from}$$

D to locate C is $\omega = 30\,[3 \times .3' - 1 \times 0^\circ\ 00.1'] = 0^\circ\ 24'$

D to locate B is $\omega = 60\,[3 \times .3' - 2 \times 0^\circ\ 00.1'] = 0^\circ\ 42'$

D to locate A is $\omega = 90\,[3 \times .3' - 3 \times 0^\circ\ 00.1'] = 0^\circ\ 54'$.

If one of the chords be fractional and the change of curvature per chord be fractional also.

$$\text{Let } \frac{n}{F} = 15' = \tfrac{1}{4},\ C = 1^\circ\ 30' = 1\tfrac{1}{2}$$

then by (61)

$$\omega = L\left[\, D\Delta \pm \left(N + \frac{n}{F}\right) C \times 0^\circ\ 00.1' \,\right].$$

To locate any point, as $G + 15$ from tangent at D, then

$$\omega = 105\,[3 \times .3 + (3 + \tfrac{1}{2}) \times 1\tfrac{1}{2} \times 0^\circ\ 00.1'] = 2^\circ\ 29'.$$

To locate any point, as $A + 15'$ from tangent at D, then

$$\omega = 75\,[3 \times .3' - (2 + \tfrac{1}{2})\ 1\tfrac{1}{2} \times 0^\circ\ 00.1'] = 0^\circ\ 39'.$$

If we have run the curve from A to D and changed the instrument to D in order to place the line of sight tangent to D, take a backsight on A and deflect $0^\circ\ 54'$ and we have a tangent at D. To facilitate the use of the tables it is best to set the vernier at $0^\circ\ 54'$ and set the telescope on line AD, turn the vernier to O and continue deflection as tabulated, reading downward from D, locating the points E, F, G, etc. If the curve is being run from D towards A then set the vernier at the angle indicated for any angle G, when backsight is on G from D deflect from zero and continue to deflect the angles tabulated in succession, reading up the column from D to locate C, B and A. The degree of the curvature at the instrument point controls the deflections either way. The above explanation enables us to run the transition curve from the point of greatest radii to that of its least radius, and vice versa.

If we take the curvature "D" at the position of the instrument as the basis of calculation, then equation (61) can be applied directly to get on tangent at the position of instrument (after backsight on the last instrument point), using the $-$ sign for running towards G and the $+$ sign for running towards A.

TABLE OF CIRCULAR ARCS

Length of Circular Arcs at Radius = 1						Decimals of a Degree			
Sec.	Length	Min.	Length	Deg.	Length	Min.	Decimal	Sec.	Decimal
1	.000005	1	.000291	1	.017453	1	.01667	1	.00028
2	.000010	2	.000582	2	.034906	2	.03333	2	.00055
3	.000015	3	.000873	3	.052360	3	.05000	3	.00083
4	.000019	4	.001164	4	.069813	4	.06667	4	.00111
5	.000024	5	.001454	5	.087266	5	.08333	5	.00139
6	.000029	6	.001745	6	.104720	6	.10000	6	.00167
7	.000034	7	.002036	7	.122173	7	.11667	7	.00195
8	.000039	8	.002327	8	.139626	8	.13333	8	.00222
9	.000044	9	.002618	9	.157080	9	.15000	9	.00250
10	.000048	10	.002909	10	.174533	10	.16667	10	.00278
11	.000053	11	.003200	11	.191986	11	.18333	11	.00306
12	.000058	12	.003491	12	.209439	12	.20000	12	.00333
13	.000063	13	.003781	13	.226893	13	.21667	13	.00361
14	.000068	14	.004072	14	.244346	14	.23333	14	.00389
15	.000073	15	.004363	15	.261799	15	.25000	15	.00417
16	.000078	16	.004654	16	.279253	16	.26667	16	.00445
17	.000083	17	.004945	17	.296706	17	.28333	17	.00477
18	.000087	18	.005236	18	.314159	18	.30000	18	.00500
19	.000092	19	.005527	19	.331613	19	.31667	19	.00528
20	.000097	20	.005818	20	.349066	20	.33333	20	.00556
21	.000102	21	.006109	21	.366519	21	.35000	21	.00583
22	.000107	22	.006399	22	.383972	22	.36667	22	.00611
23	.000111	23	.006690	23	.401426	23	.38333	23	.00639
24	.000116	24	.006981	24	.418879	24	.40000	24	.00667
25	.000121	25	.007272	25	.436332	25	.41667	25	.00695
26	.000126	26	.007563	26	.453786	26	.43333	26	.00722
27	.000131	27	.007854	27	.471239	27	.45000	27	.00750
28	.000136	28	.008145	28	.488692	28	.46667	28	.00778
29	.000141	29	.008436	29	.506145	29	.48333	29	.00806
30	.000145	30	.008727	30	.523599	30	.50000	30	.00833
31	.000150	31	.009017	31	.541052	31	.51667	31	.00861
32	.000155	32	.009308	32	.558505	32	.53333	32	.00888
33	.000160	33	.009599	33	.575959	33	.55000	33	.00916
34	.000165	34	.009890	34	.593412	34	.56667	34	.00944
35	.000170	35	.010181	35	.610865	35	.58333	35	.00972
36	.000175	36	.010472	36	.628318	36	.60000	36	.01000
37	.000179	37	.010763	37	.645772	37	.61667	37	.01028
38	.000184	38	.011054	38	.663225	38	.63333	38	.01055
39	.000189	39	.011345	39	.680678	39	.65000	39	.01083
40	.000194	40	.011635	40	.698132	40	.66667	40	.01111
41	.000199	41	.011926	41	.715585	41	.68333	41	.01139
42	.000204	42	.012217	42	.733038	42	.70000	42	.01166
43	.000209	43	.012508	43	.750492	43	.71667	43	.01194
44	.000213	44	.012800	44	.767945	44	.73333	44	.01222
45	.000218	45	.013090	45	.785398	45	.75000	45	.01250
46	.000223	46	.013381	46	.802851	46	.76667	46	.01278
47	.000228	47	.013672	47	.820305	47	.78333	47	.01306
48	.000233	48	.013963	48	.837758	48	.80000	48	.01333
49	.000238	49	.014253	49	.855211	49	.81667	49	.01361
50	.000242	50	.014544	50	.872665	50	.83333	50	.01389
51	.000247	51	.014835	51	.890118	51	.85000	51	.01417
52	.000252	52	.015126	52	.907571	52	.86667	52	.01444
53	.000257	53	.015417	53	.925024	53	.88333	53	.01472
54	.000262	54	.015708	54	.942478	54	.90000	54	.01500
55	.000267	55	.016000	55	.959931	55	.91667	55	.01528
56	.000272	56	.016290	56	.977384	56	.93333	56	.01556
57	.000276	57	.016580	57	.994838	57	.95000	57	.01583
58	.000281	58	.016871	58	1.012291	58	.96667	58	.01611
59	.000286	59	.017162	59	1.029744	59	.98333	59	.01639
60	.000291	60	.017453	60	1.047198	60	1.00000	60	.01667

Spiral I. P = 286,500, Subchord 50 Feet for One Degree Change of Curvature. g = 4.7 ft.
BY RECTANGULAR CO-ORDINATES. i = 480.

$\frac{P}{r}$	Deg. of Curv.	$\frac{L}{2r}57°18'$	$L\left(1-\frac{L^2}{40r^2}\right)$	$\frac{L^2}{6r}$	$\frac{y}{4}$	$x - r\sin a$	$\frac{f}{2}$	$\sqrt{x^2+y^2}$	$\frac{L}{2r}57.3°$	$L-(x_f+\frac{1}{2}L)$		$\frac{P}{L}$	Deg. of Curv.	$\frac{P}{r}$
L	D	$a°$	x	y	f	x_f	y_f	$l.c.$	$a°$	d		r	D	L
0	0° A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0° A	0
25	30′	4½′	25.00	.01	.00	12.50	.00	25.00	0.07°	.....	.....	11460.0	30′	25
50	1° B	15′	50.00	.07	.02	25.00	.01	50.00	0.25°	.....	.....	5730.0	1° B	50
75	1°30′	34′	75.00	.24	.06	37.50	.03	75.00	0.57°	.....	.....	3820.0	1°30′	75
100	2° C	1°00′	100.00	.58	.15	50.00	.07	100.00	1.00°	.....	.....	2865.0	2° C	100
25	2°30′	1°34′	124.99	1.14	.28	62.50	.14	125.00	1.07°	.....	.....	2292.0	2°30′	125
150	3° D	2°15′	149.98	1.96	.49	75.00	.25	149.99	2.25°	.....	.....	1910.0	3° D	150
175	3° 30′	3°04′	174.95	3.12	.78	87.50	.39	149.98	3.07°	.....	.....	1637.1	3°30′	175
200	4° E	4°00′	199.91	4.65	1.16	99.99	.58	199.97	4.00°	.01	.....	1432.5	4° E	200
225	4°30′	5°04′	224.82	7.35	1.84	112.48	.92	224.93	5.07°	.02	.....	1273.3	4°30′	225
250	5° F	6°15′	249.73	9.08	2.27	124.97	1.14	249.89	6.25°	.03	.....	1146.0	5° F	250

L	D	Transit on T'ng'nt	By Deflections toward A; $\omega = L\left[D\Delta - \left(N + \frac{n}{F}\right) C\, 0°\, 00.1'\right]$								$\delta = \frac{2}{3}\alpha$	Gauge = 4.7 ft.		
0	0° A	At A	03′	10′	23′	40′	1°03′	1°30′	2°03′	2°40′	3°22½′	4°10′	0° A	250
25	30′	01½′	+25	06′	17½′	34′	55′	1°21½′	1°2½′	2°29′	3°10′	3°56′	30′	225
50	1° B	05′	05′	B	10′	25′	45′	1°10′	1°40′	2°15′	2°55′	3°40′	1° B	200
75	1°30′	11′	12½′	08½′	+25	14′	32½′	56½′	1°25′	1°59′	2°37½′	3°21′	1°30′	175
100	2° C	20′	22′	20′	12′	C	18′	40′	1°08′	1°40′	2°17½′	3°00′	2° C	150
125	2°30′	31′	35′	33½′	27½′	16′	+25	21′	47½′	1°19′	1°55′	2°36′	2°30′	125
150	3° D	45′	50′	50′	45′	35′	20′	D	25′	55′	1°30′	2°10	3° D	100
175	3°30′	1°01′	1°07½′	1°08½′	1°05′	56′	42½′	23½′	+25	29′	1°02½′	1°41′	3°30′	75
200	4° E	1°20′	1°27′	1°30′	1°27′	1°20′	1°16′	50′	27′	E	3°2½′	1°10′	4° E	50
225	4°30′	1°41′	1°50′	1°54′	1°52½′	1°46′	1°19′	1°35′	57½′	31′	+25	36′	4°30′	25
250	5° F	2°05′	2°15′	2°20′	2°20′	2°15′	2°05′	1°50′	1°30′	1°05′	35′	at F	5° F	0
Gauge = 4.7 ft.		$\delta = \frac{\alpha}{3}$	By Deflection from A; $\omega = L\left[D\Delta + \left(N + \frac{n}{F}\right) C\, 0°\, 00.1'\right]$								Transit on T'ng'nt	D		L

Spiral II. P= 171,820, Subchord = 30 Feet for One Degree Change of Curvature. g = 4.7 ft. i = 400.

BY RECTANGULAR CO-ORDINATES.

L	*D*	α°	*x*	*y*	*f*	*xf*	*yf*	*l.c.*	α°	*d*		*r*	*D*	*L*
0	0° A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0° A	0
30	1° B	09′	30.00	.03	.01	15.00	.00	30.00	0.15°	.....	.....	5730.0	1° B	30
60	2° C	36′	60.00	.21	.05	30.00	.03	60.00°	0.60°	.....	.....	2865.0	2° C	60
75	2°30′	57′	75.00	.40	.10	37.50	.05	75.00	0.95°	.....	.....	2292.0	4°30′	45
90	3° D	1°21′	90.00	.71	.18	45.00	.09	90.00	1.35°			1910.0	3° D	90
105	3°30′	1°51′	104.99	1.12	.28	52.50	.14	105.00	1.85°			1637.1	3°30′	75
120	4° E	2°24′	119.98	1.67	.41	60.00	.21	119.99	2.40°	.....	.....	1432.5	4° E	120
135	4°30′	3°03′	134.96	2.38	.59	67.50	.29	134.98	3.05°	.....	.....	1273.3	2°30′	105
150	5° F	3°45′	149.94	3.27	.82	74.99	.41	149.97	3.75°	.01	.....	1146.0	5° F	150
165	5°30′	4°33′	164.89	4.35	1.09	82.98	1.04	164.95	4.55°	.02	.....	1041.8	1°	150
180	6° G	5°24′	179.84	5.65	1.41	89.97	.71	179.93°	5.40°	.03	.....	955.0	6° G	180

BY DEFLECTIONS.

L	D	Transit on Tangent	By Deflections toward A; ω =									δ	Gauge = 4.7 ft.		
0	0° A	At A	06′	24′	37½′	54′	1°13½′	1°36′	2°01½′	2°30′	3°01½′	3°36′	0° A	180
30	1° B	03′	B	15′	27½′	42′	1°00′	1°21′	1°45′	2°12′	2°42′	3°15′	1° B	150
60	2° C	12′	12′	C	10½′	24′	40½′	1°00′	1°22′	1°48′	2°16½′	2°48′	2° C	120
75	2°30′	19′	20′	10′	+15	13′	28½′	47′	1°09′	1°34′	2°01½′	2°32′	2°30′	105
90	3° D	27′	30′	21′	12′	D	15′	33′	54′	1°18′	1°45′	2°15′	3° D	90
105	3°30′	37′	41′	34′	25½′	14′	+15	17′	37½′	1°01′	1°27′	1°56′	3°30′	75
120	4° E	48′	54′	48′	40½′	30′	16½′	E	19½′	42′	1°07½′	1°36′	4° E	60
135	4°30′	1°01′	1°08′	1°04′	57′	47′	34½′	19′	+15	22′	46½′	1°14′	4°30′	45
150	5° F	1°15′	1°24′	1°21′	1°15′	1°06′	54′	39′	21′	F	24′	51′	5° F	30
165	5°30′	1°31′	1°41′	1°40′	1°34½′	1°26′	1°15′	1°01′	43½′	23′	+15	26′	5°30′	15
180	6° G	1°48′	2°00′	2°00′	1°55½′	1°48′	1°37½′	1°24′	1°07½′	48′	25′	At G	6° G	0
Gauge = 4.7 ft.		δ	By Deflections from A; ω =									Transit on T'ng'nt	D		L

TABLES.

Spiral III. P = 114,600, Subchord = 20 Feet for One Degree Change of Curvature. $g = 4.7$ ft. $i = 360$.

BY RECTANGULAR CO-ORDINATES.

L	D		α°	x	y	f	x_f	y_f	l. c.	α°	d		r	D		L
0	0°	A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0°	A	0
20	1°	B	06′	20.00	.01	.....	20.00	.001	20.00	0.10°	.....	.....	5730.0	1°	B	20
40	2°	C	24′	40.00	.09	.02	20.00	.01	40.00	0.40°	.....	.....	2865.0	2°	C	40
60	3°	D	54′	60.00	.31	.08	30.00	.04	.....	0.90°	.....	.....	1910.0	3°	D	60
80	4°	E	1°36′	79.99	.74	.19	40.00	.10	80.00	1.60°	.....	.....	1432.5	4°	E	80
100	5°	F	2°30′	99.98	1.45	.36	50.00	.18	100.00	2.50°	.....	.....	1146.0	5°	F	100
120	6°	G	3°36′	119.95	2.51	.63	59.99	.32	119.98	3.60°	.01	.....	955.0	6°	G	120
140	7°	H	4°54′	139.90	3.99	1.00	69.97	.50	139.95	4.90°	.02	.....	818.8	7°	H	140
160	8°	I	6°24′	159.80	5.95	1.48	79.95	.74	159.87	6.40°	.04	.....	716.3	8°	I	160
180	9°	J	8°06′	179.64	8.47	2.11	89.93	1.06	179.79	8.10°	.06	.....	636.7	9°	J	180
200	10°	K	10°00′	199.40	11.61	2.89	99.90	1.45	199.68	10.00°	.10	.....	573.0	10°	K	200

BY DEFLECTIONS.

L	D		Transit on T'ng'nt	By Deflections toward A; ω =									δ	Gauge =4.7 ft.		
0	0°	A	At A	04′	16′	36′	1°04′	1°40′	2°24′	3°16′	4°16′	5°24′	6°40′	0°	A	200
20	1°	B	02′	B	10′	28′	54′	1°28′	2°10′	3°00′	3°58′	5°04′	6°18′	1°	B	180
40	2°	C	08′	08′	C	16′	40′	1°12′	1°52′	2°40′	3°36′	4°40′	5°52′	2°	C	160
60	3°	D	18′	20′	14′	D	22′	52′	1°30′	2°16′	3°10′	4°12′	5°22′	3°	D	140
80	4°	E	32′	36′	32′	20′	E	28′	1°04′	1°48′	2°40′	3°40′	4°48′	4°	E	120
100	5°	F	50′	56′	54′	44′	26′	F	34′	1°16′	2°06′	3°04′	4°10′	5°	F	100
120	6°	G	1°12′	1°20′	1°20′	1°12′	56′	32′	G	40′	1°28′	2°24′	3°28′	6°	G	80
140	7°	H	1°38′	1°48′	1°50′	1°44′	1°30′	1°08′	38′	H	46′	1°40′	2°42′	7°	H	60
160	8°	I	2°08′	2°20′	2°24′	2°20′	2°08′	1°48′	1°20′	44′	I	52′	1°52′	8°	I	40
180	9°	J	2°42′	2°56′	3°02′	3°00′	2°50′	2°32′	2°06′	1°32′	50′	J	58′	9°	J	20
200	10°	K	3°20′	3°36′	3°44′	3°44′	3°36′	3°20′	2°56′	2°24′	1°44′	56′	at K	10°	K	0
Gauge =4.7 ft.			δ	By Deflection from A; ω =									Transit on T'ng'nt	D		L

Spiral IV. P = 85,950, Subchord = 15 Feet for One Degree Change of Curvature. g = 4.7 ft. i = 340

BY RECTANGULAR CO-ORDINATES.

L	A		a°	x	y	f	x_f	y_f	l.c.	a°	d		r	D		L
0	0°	A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0°	A	0
30	2°	C	18′	30.00	.05	.01	15.00	.00	30.00	.30°	.....	.....	2865.0	2°	C	30
60	4°	E	1°12′	60.00	.42	.11	30.00	.06	60.00	1.20°	.....	.....	1432.5	4°	E	60
75	5°	F	1°52½′	74.99	.82	.20	37.50	.10	75.00	1.875°	.....	.....	1146.0	5°	F	75
90	6°	G	2°42′	89.98	1.41	.35	45.00	.18	90.00	2.70°	.....	.....	955.0	6°	G	90
105	7°	H	3°40½′	104.96	2.24	.56	52.49	.28	104.99	3.675°	.01	.....	818.8	7°	H	105
120	8°	I	4°48′	119.92	3.35	.84	59.97	.42	119.97	4.80°	.03	.....	716.3	8°	I	120
135	9°	J	6°04½′	134.85	4.77	1.19	67.44	.60	134.95	6.075°	.06	.....	636.7	9°	J	135
150	10°	K	7°30′	149.75	6.54	1.63	74.90	.82	149.93	7.50°	.10	.....	573.0	10°	K	150
165	11°	L	9°04½′	164.59	8.70	2.17	82.35	1.09	164.86	9.075°	.15	.....	521.8	11°	L	165
180	12°	M	10°48′	179.36	11.28	2.81	89.79	1.41	179.77	10.80°	.21	.....	477.5	12°	M	180

L	D	Transit on T'ng'nt	By Deflections toward A; ω =									δ	Gauge = 4.7 ft.		
0	0° A	At A	12′	48′	1°15′	1°48′	2°27′	3°12′	4°03′	5°00′	6°03′	7°12′	0° A	180
30	2 C	06′	C	30′	54′	1°24′	2°00′	2°42′	3°30′	4°24′	5°24′	6°30′	2° C	150
60	4° E	24′	24′	E	21′	48′	1°21′	2°00′	2°45′	3°36′	4°33′	5°36′	4° E	120
75	5° F	37½′	40½′	19½′	F	25½′	57′	1°34½′	2°18′	3°07½′	4°03′	5°04½′	5° F	105
90	6° G	54′	1°00′	42′	24′	G	30′	1°06′	1°48′	2°36′	3°30′	4°30′	6° G	90
105	7° H	1°13½′	1°22½′	1°07½′	51′	28½′	H	34½′	1°15′	2°01½′	2°54′	3°52½′	7° H	75
120	8° I	1°36′	1°48′	1°36′	1°21′	1°00′	33′	I	39′	1°24′	2°15′	3°12′	8° I	60
135	9° J	2°01½′	2°16½′	2°07½′	1°54′	1°34½′	1°09′	37½′	J	43½′	1°33′	2°28½′	9° J	45
150	10° K	2°30′	2°48′	2°42′	2°30′	2°12′	1°48′	1°18′	42′	K	48′	1°42′	10° K	30
165	11° L	3°01½′	3°22½′	3°19½′	3°09′	2°52½′	2°30′	2°01½′	1°27′	46½′	L	52½′	11° L	15
180	12° M	3°36′	4°00′	4°00′	3°51′	3°36′	3°15′	2°48′	2°15′	1°36′	51′	at M	12° M	0
Gauge = 4.7 ft.		δ	By Deflection from A; ω =									Transit on T'ng'nt	D		L

TABLES.

Spiral V. P = 57,300, Subchord = 10 Feet for One Degree Change of Curvature. g = 4.7 ft. i = 320

BY RECTANGULAR CO-ORDINATES.

L	D		a°	x	y	f	x_f	y_f	l. c.	a°	d		r	D		L
0	0°	A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0°	A	0
20	2°	C	12′	20.00	02	.01	10.00	.00	20.00	0.20°	.....	.....	2865.0	2°	C	20
40	4°	E	48′	40.00	.19	.05	20.00	.03	40.00	0.80°	.....	.....	1432.5	4°	E	40
60	6°	G	1°48′	59.99	.63	.16	30.00	.08	60.00	1.80°	.....	.....	955.0	6°	G	60
80	8°	I	3°12′	79.98	1.49	.37	40.00	.19	79.99	3.20°	.....	.....	716.3	8°	I	80
100	10°	K	5°00′	99.93	2.90	.72	49.99	.36	99.97	5.00°	.....	.....	573.0	10°	K	100
120	120	M	7°12′	119.82	5.02	1.25	59.97	.63	119.93	7.20°	.03	.....	477.5	12°	M	120
130	13°	N	8°27′	129.74	6.38	1.59	64.96	.80	129.90	8.45°	.04	.....	440.8	13°	N	130
140	14°	O	9°48′	139.62	7.97	1.98	69.95	.99	137.85	9.80°	.05	.....	409.3	14°	O	140
150	15°	P	11°15′	149.47	9.80	2.44	79.94	1.22	149.79	11.25°	.06	.....	382.0	15°	P	150
160	16°	Q	12°48′	159.26	11.88	2.95	79.93	1.48	159.70	12.80°	.07	.....	358.1	16°	Q	160

L	D		Transit on T'ng'nt	By Deflections toward A; ω =									δ	Gauge =4.7 ft.		
0	0°	A	At A	08′	32′	1°12′	2°08′	3°20′	4°48′	5°38′	6°32′	7°30′	8°32′	0°	A	160
2	2°	C	04′	C	20′	56′	1°48′	2°56′	4°20′	5°08′	6°00′	6°56′	7°56′	2°	C	140
4	4°	E	16′	16′	E	32′	1°20′	2°24′	3°44′	4°30′	5°20′	6°14′	7°12′	4°	E	120
6	6°	G	36′	40′	28′	G	44′	1°44′	3°00′	3°44′	4°32′	5°24′	6°20′	6°	G	100
8	8°	I	1°04′	1°12′	1°04′	40′	I	56′	2°08′	2°50′	3°36′	4°26′	5°20′	8°	I	80
10	10°	K	1°40′	1°52′	1°48′	1°28′	52′	K	1°08′	1°48′	2°32′	3°20′	4°12′	10°	K	60
12	12°	M	2°24′	2°40′	2°40′	2°24′	1°52′	1°04′	M	38′	1°20′	2°06′	2°56′	12°	M	40
13	13°	N	2°49′	3°07′	3°09′	2°55′	2°25′	1°39′	37′	N	41′	1°26′	2°15′	13°	N	30
14	14°	O	3°16′	3°36′	3°40′	3°28′	3°00′	2°16′	1°16′	40′	O	44′	1°32′	14°	O	20
15	15°	P	3°45′	4°07′	4°13′	4°03′	3°37′	2°55′	1°57′	1°22′	43′	P	47′	15°	P	10
16	16°	Q	4°16′	4°40′	4°48′	4°40′	4°16′	3°36′	2°40′	2°06′	1°28′	46′	At Q	16°	Q	0
Gauge =4.7 ft.			δ	By Deflections from A; ω =									Transit on T'ng'nt	D		L

TABLES.

Spiral VI. P = 28,650, Subchord = 10 Feet for Two Degrees Change of Curvature. g = 4.7 ft. i = 300.

BY RECTANGULAR CO-ORDINATES.

L	D		a°	x	y	f	x_f	y_f	l. c.	a°	d		r	D		L
0	0°	A	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	0°	A	140
20	4°	C	24′	20.00	.05	.01	10.00	.00	20.00	0.40°	.....	.....	1432.5	4°	C	120
40	8°	E	1°36′	40.00	.37	.09	20.00	.05	40.00	1.60°	.....	.....	716.3	8°	E	100
60	12°	G	3°36′	59.98	1.26	.32	30.00	.16	59.99	3.60°	.....	.....	477.5	12°	G	80
80	16°	I	6°24′	79.90	2.98	.75	39.99	.37	79.98	6.40°	.01	.....	358.1	16°	I	60
90	18°	J	8°06′	89.80	4.24	1.06	44.97	.53	89.92	8.10°	.03	.....	318.3	18°	J	50
100	20°	K	10°00′	99.70	5.81	1.45	49.95	.72	99.90	10.00°	.05	.....	286.5	20°	K	40
110	22°	L	12°06′	109.50	7.74	1.93	54.92	.96	109.81	12.10°	.08	.....	260.5	22°	L	30
120	24°	M	14°24′	119.20	10.01	2.51	59.88	1.25	119.45	14.40°	.12	.....	238.8	24°	M	20
130	26°	N	16°54′	128.90	12.70	3.18	64.80	1.58	129.25	16.90°	.20	.....	220.4	26°	N	10
140	28°	O	19°35′	138.40	15.83	3.97	69.70	1.97	139.30	19.60°	.30	.....	204.6	28°	O	0

L	D		Transit on T'ng'nt	By Deflections toward A; ω =									δ	Gauge =4.7 ft.		
0	0°	A	At A	16′	1°04′	2°24′	4°16′	5°24′	6°40′	8°04′	9°36′	11°16′	13°04′	0°	A	140
20	4°	C	08′	C	40′	1°52′	3°36′	4°40′	5°52′	7°12′	8°40′	10°16′	12°00′	4°	C	120
40	8°	E	32′	32′	E	1°04′	2°40′	3°40′	4°48′	6°04′	7°28′	9°00′	10°40′	8°	E	100
60	12°	G	1°12′	1°20′	56′	G	1°28′	2°24′	3°28′	4°40′	6°00′	7°28′	9°04′	12°	G	80
80	16°	I	2°08′	2°24′	2°08′	1°20′	I	52′	1°52′	3°00′	4°16′	5°40′	7°12′	16°	I	60
90	18°	J	2°42′	3°02′	2°50′	2°06′	50′	J	58′	2°04′	3°18′	4°40′	6°10′	18°	J	50
100	20°	K	3°20′	3°44′	3°36′	2°56′	1°44′	56′	K	1°04′	2°16′	3°36′	5°04′	20°	K	40
110	22°	L	4°02′	4°20′	4°26′	3°50′	2°42′	1°56′	1°02′	L	1°10′	2°28′	3°54′	22°	L	30
120	24°	M	4°48′	5°20′	5°20′	4°48′	3°44′	3°00′	2°08′	1°08′	M	1°16′	2°40′	24°	M	20
130	26°		5°38′	6°14′	6°18′	5°50′	4°50′	4°08′	3°18′	2°20′	1°14′	N	1°22′	26°	N	10
140	28°	O	6°32′	7°12′	7°20′	6°56′	6°00′	5°20′	4°32′	3°36′	2°32′	1°20′	at O	28°	O	0
Gauge =4.7 ft.			δ	By Deflection from A; ω =									Transit on T'ng'nt	D		L

Spiral VII. P = 1719, Subchord = $\frac{6}{10}$ Feet for Two Degrees Change of Curvature. i = 100.

BY RECTANGULAR CO-ORDINATES.

L	D	Sta.	a°	x	y	f	x	y_f	l. c.	a°	d	r	D	Sta.	L
0.															0.
3.	10°	1	0°09′	3.00	.003	.000	1.50	.00	3.00	0.15°	...	573.0	10°	1	3.
6.	20°	2	0°36′	6.00	.02	.005	3.00	.00	6.00	0.60°	...	286.5	20°	2	6.
9.	30°	3	1°21′	9.00	.07	.02	4.50	.01	9.00	1.35°	...	191.0	30°	3	9.
10.5	35°	4	1°50′	10.50	.11	.03	5.25	.01	10.50	1.83°	...	163.7	35°	4	10.5
12.	40°	5	2°24′	12.00	.17	.04	6.00	.02	12.00	1.40°	...	143.2	40°	5	12.
13.5	45°	6	3°02′	13.50	.24	.06	6.75	.03	13.50	3.03°	...	127.3	45°	6	13.5
15.	50°	7	3°45′	15.00	.33	.08	7.50	.04	15.00	3.75°	...	114.6	50°	7	15.
16.5	55°	8	4°32′	16.49	.44	.11	8.25	.05	16.50	4.53°	...	104.2	55°	8	16.5
18.	60°	9	5°24′	17.99	.56	.14	9.00	.07	17.00	5.40°	...	95.5	60°	9	18.
18.9	63°	10	5°57′	18.88	.66	.15	9.45	.07	18.89	5.95°	...	90.9	63°	10	18.9
19.8	66°	11	6°32′	19.78	.76	.19	9.90	.09	19.79	6.53°	...	86.8	66°	11	19.8
20.7	69°	12	7°09′	20.67	.86	.22	10.35	.11	20.68	7.15°	...	83.0	69°	12	20.7
21.6	72°	13	7°46′	21.56	.98	.24	10.80	.12	21.58	7.77°	...	79.7	72°	13	21.6
22.5	75°	14	8°26′	22.45	1.10	.25	11.25	.12	22.48	8.43°	...	76.4	75°	14	22.5
23.4	78°	15	9°08′	23.34	1.24	.31	11.70	.15	23.37	9.13°	...	73.4	78°	15	23.4
24.3	81°	16	9°46′	24.23	1.39	.35	12.15	.17	24.27	9.77°	...	70.7	81°	16	24.3
25.2	84°	17	10°35′	25.12	1.55	.39	12.60	.19	25.16	10.58°	...	68.2	84°	17	25.2
26.1	87°	18	11°21′	26.00	1.73	.43	13.04	.21	26.06	11.35°	.01	65.9	87°	18	26.1
27.	90°	19	12°12′	26.88	1.89	.48	13.49	.24	26.79	12.20°	.01	63.7	90°	19	27.
27.6	92°	20	12°42′	27.46	2.03	.51	13.78	.25	27.53	12.70°	.02	62.3	92°	20	27.6
28.2	94°	21	13°16′	28.05	2.18	.54	14.08	.27	28.13	13.27°	.02	60.9	94°	21	28.2
28.8	96°	22	13°49′	28.63	2.30	.58	14.37	.29	28.72	13.82°	.03	59.7	96°	22	28.8
29.4	98°	23	14°24′	29.22	2.43	.62	14.66	.31	29.32	14.40°	.04	58.4	98°	23	29.4
30.	100°	24	15°00′	29.80	2.60	.65	14.96	.32	29.91	15.00°	.04	57.3	100°	24	30.
30.6	102°	25	15°36′	30.38	2.76	.69	15.25	.34	30.50	15.60°	.05	56.2	102°	25	30.6
31.2	104°	26	16°13′	30.96	2.92	.73	15.55	.37	31.10	16.22°	.05	55.1	104°	26	31.2
31.8	106°	27	16°51′	31.54	3.09	.77	15.84	.39	31.69	16.85°	.06	54.1	106°	27	31.8
32.4	108°	28	17°29′	32.11	3.27	.81	16.13	.41	32.28	17.48°	.07	53.1	108°	28	32.4
33.	110°	29	18°05′	32.68	3.45	.86	16.42	.43	32.87	18.08°	.08	52.1	110°	29	33.
33.6	112°	30	18°50′	33.27	3.64	.91	16.71	.46	33.47	18.83°	.09	51.1	112°	30	33.6
34.2	114°	31	19°29′	33.85	3.84	.96	17.00	.48	34.06	19.48°	.10	50.3	114°	31	34.2

APPENDIX I.

PROBLEM.

Given: The altitude of two circumpolar stars, nearly opposite each other, the local time of Meridian passage of either star, their difference in time of Meridian passage and the polar distance of each star,

TO FIND THE LATITUDE, THE LOCAL TIME AND THE MERIDIAN OF THE PLACE.

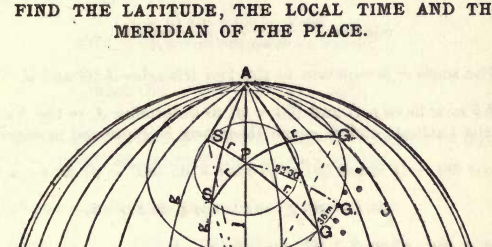


Fig. 11.

For convenience, let Polaris and Alioth be the stars observed; O the position of the observer and center of the celestial sphere. $ONEW$ be the north half of that portion of the celestial sphere above the horizon WNE, with A as zenith.

Let $GC = c$ or $G_{,,}C_{,,} = c_{,,}$ be the altitude of Alioth at the time of observation and $BS = g$ or $B_{,,}S_{,,} = g_{,,}$ the altitude of Polaris * 35 minutes later; then the plane passing through the points S, G and O will pass through the pole P and cut the celestial sphere in the line $r + r_{,} SG$ or $S_{,}G_{,,}$, r † and $r_{,}$ being the polar distance respectively of Polaris and Alioth and $AP = s$ the co-latitude of the place O. Then the plane ASO or $AS_{,,}O$ will cut the side of the

---

* At this date.

† The position of these stars have a slight annual change with reference to the pole, which must be taken into account from year to year.

spherical triangle $AS = AB - BS = (90 - g)$ or $AS_{II} = AB_{II} - B_{II}S_{II}$ from the celestial sphere. Similarly, AGO or $AG_{II}O$ will cut $(90 - c)$ or $(90 - c_{II})$. The sides f, $r + r_{I}$ and $(90 - c)$ or $(90 - c_{II})$ form the spherical triangle ASG or $AS_{II}G_{II}$, in which the sides are known; from which we find the angle S or S_{II} by the spherical formula:

$$\cos ASG = \cos S = \frac{(\cos 90 - c) - \cos (90 - g) \cos (r + r)}{\sin (90 - g) \sin (r + r_1)},$$

but $\cos (90 - c) = \sin c$ and $\cos (90 - g) = \sin g$, hence:

$$\cos S = \frac{\sin c - \sin g \cos (r + r_{I})}{\cos g \sin (r + r_{I})}.$$

The angle S is common to the two triangles ASG and ASP.

We now have r, S and $(90 - g)$ to determine A = the Azimuth, l = the Latitude and P = the Hour Ang e (expressed in degrees).

$$\cos (90 - l) = \cos (90 - g) \cos r \pm \sin (90° - g) \sin r \cos S,$$

or
$$\sin l = \sin g \cos r \pm \cos g \sin r \cos S, \qquad (a)$$

using $-$ sign when $S > 90$; and since

$$\sin (90 - l) : \sin (90 - g) :: \sin S : \sin P,$$

or
$$\cos l : \cos g :: \sin S : \sin P$$

$$\sin P = \frac{\cos g \sin S}{\cos l}. \qquad (b)$$

P represents the angle the plane OSP makes with the plane OPA at the time of observation on S and $P° \times 4$ the Hour Angle, expressed in minutes of sidereal time, from the plane $ONPA$. Also in the triangle ASP we have

$$\sin A = \frac{\sin r \sin P}{\sin (90 - g)} = \frac{\sin r \sin P}{\cos g} \qquad (c)$$

in which A is the angle between the planes ABO and ANO, which is the Azimuth Angle for the Meridian Plane ANO.

A similar solution applies to the triangles $AS_{II}P$ and $AS_{II}G_{II}$. Equations (a), (b) and (c) solve the problem.

When the altitude of Alioth is much in excess of that of Polaris, the observations are not easily made in higher latitudes with the ordinary engineer's transit, unless equipped with a prismatic eyepiece.

Sidereal hours $\times$ .9972696 = mean solar hours.

EXAMPLE. Given $g = 40$; $c = 20$; $r = 33° 30'$ to find the latitude l. We first find the angle S by the formula,

$$\cos S = \frac{\sin c - \sin g \cos (r + r_1)}{\cos g \sin (r + r)} = \frac{.3292 - .64279 \times .82181}{.76604 \times .56976}$$

$$= -\frac{.18623}{.43646} = -.42667 = \cos 115.15.$$

To find the latitude we have (a)

$$\sin l = \sin g \cos r \pm \cos g \sin r \cos S$$

using the $-$ sign (since $S > 90°$),

$\sin l = .64279\ .9998 - .76604 \times .0215 \times 42667 = .642661 - 00702$
$= .63564 = \sin 39° 28'$, whence the Latitude $39° 28'$.

To find the local time we have (b)

$$\sin P = \frac{\cos g \sin S}{\cos r} = \frac{.76604 \quad .90445}{.7720} = .89741 = \sin 63° 49', P \times 4^m$$

$= 63° 49' \times 4 = 4^h 15^m = 255$ sidereal minutes; $255 \times .99727 = 254$ mean solar minutes, or $P = 4^h 14^m$.

The time of observation was at 1 o'clock A.M., Dec. 1st, 1900, i.e. $13^h - 4^h 14^m = 8^h 46^m$ the time of meridian passage by the clock.

The local time of meridian passage was $8^h 40^m$ i.e. $8^h 46^m - 8^h 40^m = 0^h 06^m$ i.e. the clock was 6^m fast.

To find the Azimuth Angle $SAP = BON = A$, we have (c)

$$\sin A = \frac{\sin r \sin P}{\cos g} = \frac{.0215 \times .7974}{.76604} = .02518 \text{ whence } A = 1° 27'$$

SAN FRANCISCO, 1898.

Local Mean Solar Time of Meridian Passage of the North Star, Above the Pole, for Longitude 31° (2h 04m) West of Washington.*

Month	Year 1904	Year 1905	Year 1906	Year 1907	Year 1908	Year 1909	Year 1910	Dif. for One Day
	H. M.	H. M.	H. M.	H. M.	H. M.	H. M.	H. M.	M.
Jan. 1.....	6 43.9 P.M.	6 41.4 P.M.	6 42.8 P.M.	6 44.3 P.M.	6 45.7 P.M.	6 43.2 P.M.	6 44.7 P.M.	3.95
Feb. 1.....	4 41.5 "	4 39.0 "	4 40.4 "	4 41.3 "	4 43.3 "	4 40.8 "	4 42.3 "	3.95
Mar. 1.....	2 47.0 "	2 48.5 "	2 49.9 "	2 51.4 "	2 48.9 "	2 50.3 "	2 51.8 "	3.94
Apr. 1.....	0 45.0 "	0 46.5 "	0 47.9 "	0 49.4 "	0 46.8 "	0 48.3 "	0 49.8 "	3.94
May 1.....	10 43.2 A.M.	10 44.7 A.M.	10 46.1 A.M.	10 47.6 A.M.	10 45.1 A.M.	10 46.5 A.M.	10 48.0 A.M.	3.93
June 1.....	8 41.6 "	8 43.1 "	8 44.5 "	8 46.0 "	8 43.5 "	8 44.9 "	8 46.4 "	3.92
July 1.....	6 44.2 "	6 45.7 "	6 47.1 "	6 48.6 "	6 46.1 "	6 47.5 "	6 49.0 "	3.91
Aug. 1.....	4 42.8 "	4 44.3 "	4 45.7 "	4 47.2 "	4 44.7 "	4 46.1 "	4 47.6 "	3.92
Sept. 1.....	2 41.4 "	2 42.9 "	2 44.3 "	2 45.8 "	2 43.3 "	2 44.7 "	2 46.2 "	3.92
Oct. 1.....	0 43.7 "	0 45.2 "	0 46.6 "	0 48.1 "	0 45.6 "	0 47.0 "	0 48.5 "	3.93
Nov. 1.....	10 41.8 P.M.	10 43.3 P.M.	10 44.7 P.M.	10 46.2 P.M.	10 43.7 P.M.	10 45.1 P.M.	10 46.6 P.M.	3.93
Dec. 1.....	8 43.7 "	8 45.2 "	8 46.6 "	8 48.1 "	8 45.6 "	8 47.0 "	8 48.5 "	3.94

* For Eastern elongation, the mean solar time is 5 h. 59.16 m. earlier than that of Meridian passage above the pole. For Western elongation, the mean solar time is 5 h. 59.16 m. later than that of Meridian passage above the pole. For Meridian passage below the pole, the mean solar time is 11 h. 58.32 m. earlier or later than Meridian passage above the pole.

METHOD OF COMPUTING RADIUS.

The usual method of computing the radius of curvature of circular curves used in railroad location, is by the formula

$$r = \frac{\frac{1}{2}c}{\sin \frac{1}{2}D}, \tag{1}$$

in which c represents a chord of 50 feet in length, and D is the degree of the curve or central angle for a chord of 100 feet. By this method the radius does not vary inversely as the degree of the curve; while in the method of this book it is assumed to do so, and r is assumed to equal the radius of a one-degree curve divided by the degree of the proposed curve, or

$$r = \frac{5730}{D}. \tag{2}$$

These formulae give the same result for a five-degree curve for 50-foot chords, thus:

$$r = \frac{\frac{1}{2}c}{\sin \frac{1}{2}D} = \frac{25}{.021815} = 1146.00 \text{ and } r = \frac{5730}{5} = 1146.00,$$

from which we see, by the method of this book, the radius of a five-degree curve is the same computed by either formula. For less than 5° rate of curvature r eq. (1) $< r$ eq. (2) and for a rate of curvature greater than 5° r eq. (1) $> r$ eq. (2) for 50-foot stations.

Since the length of the chords is usually less than 50 feet, in the spiral the radius of curvature is assumed to vary inversely as the degree of the curve, and the same radius may be used for the main curve, by making its chords less than 50 feet, the proper length to be determined, by computation, and tabulated for field use. It is scarcely worth notice (= .002 per degree of curve less than 50 feet for each degree in rate of curvature per 100 feet of arc).

TO DETERMINE THE NUMERICAL CONSTANT IN FORMULA (60):

Place $0°00.1' = k$, then we have $\delta' = LNCk$

or
$$k = \frac{\delta'}{LNC}. \tag{1}$$

We find δ' by dividing eq. (21) by eq. (26), whence,

$$\tan \delta = \frac{y}{x} = \frac{\frac{L^2}{6r}\left(1 - \frac{L^2}{56r^2}\right)}{L\left(1 - \frac{L^2}{40r^2}\right)} +, \text{ or } \tan \delta = \frac{\frac{L}{6r}\left(1 - \frac{L^2}{56r^2} +\right)}{1 - \frac{L^2}{40r^2} +}.$$

δ should be reduced to minutes (′) before entering the formula for finding the value of k.

The above value of k is for running in a direction from A towards B, Fig. 3, or D towards E, etc.

For running in a direction from G towards F, or D towards C:

Having α' (minutes), the total number of minutes in the central angle of any given length L of the spiral, corresponding to δ' in the above equation, if we substitute $(\alpha' - \delta')$ in eq. (1), of this supplement, for δ and $2L$ for L, then calling the numerical coefficient k', we have

$$k_{,} = \frac{\alpha' - \delta' n}{2LNC} \qquad (2)$$

for running in a contrary direction from the same points as in eq. (1) of this supplement. $k_{,}$ should always come out greater than k.

The value of $k_{,}$ will be found to increase and k decrease, slightly, with α. For $L = 200$ feet, or less, and $\alpha = 10°$, or less, we may make $k_{,} = k = 0°00.1'$.

It will be seen by a trial example that the difference between this method and that of eq. (61) for a curve with $L = 400$ (with change point at 200 feet) and $\alpha = 16°$, is 0°00′03″.

The difference increases with the central angle α. The above method, with change points 200 to 300 feet apart, is quite accurate and best for preparing a set of tables, though not so easily applied in field computations as eqs. (60) or (61).

By assuming "change points" from 100 to 250 feet apart the values in the deflection tables of this book may be substituted for the second term in eq. (61) and the spiral extended indefinitely.

The following table shows the results of computations by the above formulae.

The value of the constant C in each particular case is determined by $\frac{D_{,}}{N} = C$, in which $D_{,}$ represents the degree, or rate of curvature, at the point to be located; this substituted in eq. (60b) reduces it to the form of $\delta' = LD_{,}k$. The above deduced value of C should always be applied in eq. (60) or (60b).

k is not a constant, but for small deflection and central angles its successive differences are inappreciable, but for large central angles these differences may not be disregarded.

If δ' is constant

$$k = \frac{\delta'}{LD'};\ L = \frac{\delta'}{kD'};\ D_{,} = \frac{\delta'}{kL},$$

and for $k_{,} = \frac{\alpha' - \delta'}{2LD_{,}};\ L = \frac{\alpha' - \delta'}{2k_{,}D_{,}};\ D_{,} = \frac{\alpha' - \delta'}{2k_{,}L}$

P = 114,600.

D	L	a	x	y	$\tan \frac{y}{x}$	δ	$a - \delta$	k	k'
2	40	24′	40.00	.09	0.00250	08.5′	16.5′	.1000′	.1000′
4	80	1°36′	79.99	.74	0.00925	31.8′	1°04.2′	.1000′	.1000′
6	120	3°36′	119.95	2.51	0.02092	1°12′	2°24′	.1000′	.1000′
8	160	6°24′	159.80	5.95	0.03723	2°08′	4°16′	.1000′	.1000′
10	200	10°00′	199.40	11.61	0.05822	3°20′	6°40′	.1000′	.1000′
12	240	14°24′	238.5	20°01′	0.08390	4°47.75′	9°36.25′	.09996′	.10003′
14	280	19°36′	276.7	31.66	0.11442	6°31.67′	13°04.33′	.09992′	.10006′
16	320	25°36′	313.7	46.99	0.14979	8°31′	17°05′	.09990′	.10010′
18	360	32°24′	348.7	66.32	0.19073	10°48′	21°36′	.09971′	.10012′
20	400	40°00′	380.9	89.89	0.23599	13°16.67′	26°43.33′	.09958′	.10021′
22	440	48°24′	409.6	117.72	0.28735	16°02′	32°22′	.09921′	.10047′

For the same value of δ, L and D, are inverse functions of each other (nearly), and k and k, remain the same and vary only with the deflection angles δ' and $a' - \delta'$.

in which L is in feet, $D_{\prime}$ in degrees (°), δ' in minutes; m being determined by whatever value is assumed for L and $D° = \frac{5730}{r_{\prime\prime}}$ and substituted in eq. (37) or (38).

SUPPLEMENT TO PROBLEM I.

The total intersection angle is

$$I = a + a + \iota.$$

ι is the angle due to the circular portion of the curve or $B_{\prime}B_{\prime\prime\prime}$, with fixed radius $r_{\prime\prime}$ and

$$B_{\prime}DB_{\prime\prime\prime} = I - (a_{\prime} + a),$$

whence, by making the second term in eq. (61) = 0, we have:

$\omega = LD\Delta$ = the deflection from a tangent at any point of the circular curve, as $B_{\prime}$ or $B_{\prime\prime\prime}$ to locate any other point of the circular curve between $B_{\prime}$ and $B_{\prime\prime\prime}$ inclusive of the last point $B_{\prime}$ or $B_{\prime\prime\prime}$ as the case may be.

SUPPLEMENT TO PROBLEM III.

ι = the arc of that portion of the curve, between spirals, that is *circular* and laid out by the method of circular curves, the deflection from tangent being computed by eq. (61), in which the second term, in brackets, is made = 0, whence, it becomes $\omega = LD\Delta$. ι will have to be reduced from arc to degrees and minutes to determine the total deflection.

THE PROCESS OF FIELD WORK.

Beginning at $O_{\prime}$, as instrument point, deflect from tangent $O_{\prime}X$ for successive stations by formula (60) including $B_{\prime}$ $\left(= \frac{a}{3}\right)$, then, placing the instrument at $B_{\prime}$, with backsight on $O_{\prime}$, deflect

$$\delta' = 2LNC \times 0° 00.1' \left(= \frac{2}{3} a\right).$$

The line of sight will then be a tangent common to both the spiral $O_{\prime}B_{\prime}$ and the circular curve $B_{\prime}B_{\prime\prime\prime}$ at $B_{\prime}$. Deflect thence by formula

$$\omega = LD\Delta$$

to locate stations on the circular curve $B_{\prime}B_{\prime\prime\prime}$ fixing $B_{\prime\prime\prime}$.

Set up the instrument at $B_{\prime\prime\prime}$ and, with backsight on $B_{\prime}$, deflect $\omega = LD\Delta$ for tangent at $B_{\prime\prime\prime}$, thence locate points of spiral $B_{\prime\prime\prime}O_{\prime\prime\prime}$ by applying formula (61) with − sign or by (54), any deflection

$$\delta' = 2LNC \times 0° 00.1'.$$

Having located O_{III}, set up instrument at O_{III} and with backsight on B_{III} deflect

$$\delta' = LNC \times 0° 00.1',$$

and the line of sight should coincide in direction and position with the tangent $O_{III}X$, or the semi-tangent $O_{III}d_I$.

Some prefer to run the spirals first, i.e., from O_I to B_I and from O_{III} to B_{III}, and the circular portion B_IB_{III} last, thus throwing the closing error, if any, at B_I or B_{III}.

Undoubtedly this makes a better adjustment when the points O_I and O_{III} have been fixed previously by semi-tangents.

All change points should be established by *double centers* to eliminate errors of adjustment in the transit instrument.

TRIGONOMETRIC TABLES

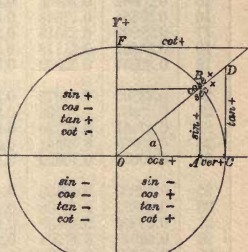
The trigonometric functions of any angle intermediate those given in the tables may be found by interpolation, thus:

What is the natural tangent of 12°43'?

From the Table,	tan. 12°50' =	0.22781
" " "	tan. 12°40' =	0.22475
Diff. for 0°10'	=	0.00306
		.3
" " 0°03'	=	0.000918*
Add " 12°40'	=	0.22475
Hence for	tan. 12°43' =	0.22567

What is the natural cotangent of 12°43'?

From the Table,	cot. 12°40' =	4.44942
" " "	cot. 12°50' =	4.38969
Diff. for 0°10'	=	0.05973
		.3
" " 0°03'	=	0.017919*
Subtract from	cot. 12°40' =	4.44942
Hence	cot. 12°43' =	4.43150



To obtain functions not given in the tables:

Vers. $a = 1 - \cos. a$; External Sec. $a = 1 - \text{Sec. } a$.

*The computation being *additive or subtractive* according as the function increases or decreases with the increase of the angle a.

Degrees	SINE 0′	10′	20′	30′	40′	50′	60′	
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89
1	0.01745	0.02036	0.02327	0 02618	0.02908	0.03199	0.03490	88
2	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87
3	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86
4	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85
5	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84
6	0.10453	0.10742	0.11031	0.11320	0.11609	0.11898	0.12187	83
7	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82
8	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81
9	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80
10	0.17365	0.17651	0 17937	0.18224	0.18500	0.18795	0.19081	79
11	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78
12	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77
13	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76
14	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75
15	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74
16	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73
17	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72
18	0.30902	0.31178	0.31454	0.31730	0.32006	0.32282	0.32557	71
19	0.32557	0.32832	0.33106	0.33381	0.33655	0.33929	0.34202	70
20	0.34202	0.34475	0.34748	0.35021	0.35293	0.35565	0.35837	69
21	0.35837	0.36108	0.36379	0.36650	0.36921	0.37191	0.37461	68
22	0.37461	0.37730	0.37999	0.38268	0.38537	0.38805	0.39073	67
23	0.39073	0.39341	0.39608	0.39875	0.40142	0.40408	0.40674	66
24	0.40674	0.40939	0.41204	0.41469	0.41734	0.41998	0.42262	65
25	0.42262	0.42525	0.42788	0.43051	0.43318	0.43575	0.43837	64
26	0.43837	0.44098	0.44359	0.44620	0.44880	0.45140	0.45399	63
27	0.45399	0.45658	0.45917	0.46175	0.46433	0.46690	0.46947	62
28	0.46947	0.47204	0.47460	0.47716	0.47971	0.48226	0.48481	61
29	0.48481	0.48735	0.48989	0.49242	0.49495	0.49748	0.50000	60
30	0.50000	0.50252	0.50503	0.50754	0.51004	0.51254	0.51504	59
31	0.51504	0.51753	0.52002	0.52250	0.52498	0.52745	0.52992	58
32	0.52992	0.53238	0.53484	0.53730	0.53975	0.54220	0.54464	57
33	0.54464	0.54708	0.54951	0.55197	0.55436	0.55678	0.55919	56
34	0.55919	0.56160	0.56401	0.56641	0.56880	0.57119	0.57358	55
35	0.57358	0.57596	0.57833	0.58070	0.58307	0.58543	0.58779	54
36	0.58779	0.59014	0.59248	0.59482	0.59716	0.59949	0.60182	53
37	0.60182	0.60414	0.60645	0.60876	0.61107	0.61337	0.61566	52
38	0.61566	0.61795	0.62024	0.62251	0.62479	0.62706	0.62932	51
39	0.62932	0.63158	0.63383	0.63608	0.63832	0.64056	0.64279	50
40	0.64279	0.64501	0.64723	0.64945	0.65166	0.65386	0.65606	49
41	0.65606	0.65825	0.66044	0.66262	0.66480	0.66697	0.66913	48
42	0.66913	0.67129	0.67344	0.67559	0.67773	0.67987	0.68200	47
43	0.68200	0.68412	0.68624	0.68835	0.69046	0.69256	0.69466	46
44	0.69466	0.69675	0.69883	0.70091	0.70298	0.70505	0.70711	45
	60′	50′	40′	30′	20′	10′	0′	Degrees
	COSINE							

Degrees	COSINE							
	0′	10′	20′	30′	40′	50′	60′	
0	1.00000	1.00000	0.99998	0.99996	0.99993	0.99989	0.99985	89
1	0.99985	0.99979	0.99973	0.99966	0.99958	0.99949	0.99939	88
2	0.99939	0.99929	0.99917	0.99905	0.99892	0.99878	0.99863	87
3	0.99863	0.99847	0.99831	0.99813	0.99795	0.99776	0.99756	86
4	0.99756	0.99736	0.99714	0.99692	0.99668	0.99644	0.99619	85
5	0.99619	0.99594	0.99567	0.99540	0.99511	0.99482	0.99452	84
6	0.99452	0.99421	0.99390	0.99357	0.99324	0.99290	0.99255	83
7	0.99255	0.99219	0.99182	0.99144	0.99106	0.99067	0.99027	82
8	0.99027	0.98986	0.98944	0.98902	0.98858	0.98814	0.98769	81
9	0.98769	0.98723	0.98676	0.98629	0.98580	0.98531	0.98481	80
10	0.98481	0.98430	0.98378	0.98325	0.98272	6.98218	0.98163	79
11	0.98163	0.98107	0.98050	0.97992	0.97934	0.97875	0.97815	78
12	0.97815	0.97754	0.97692	0.97630	0.97566	0.97502	0.97437	77
13	0.97437	0.97371	0.97604	0.97237	0.97169	0.97100	0.97030	76
14	0.97030	0.96959	0.96887	0.96815	0.96742	0.96667	0.96593	75
15	0.96593	0.96517	0.96440	0.96363	0.96285	0.96206	0.96126	74
16	0.96126	0.96046	0.95964	0.95882	0.95799	0.95715	0.95630	73
17	0.95630	0.95545	0.95459	0.95372	0.95284	0.95195	0.95106	72
18	0.95106	0.95015	0.94924	0.94832	0.94740	0.94646	0.94552	71
19	0.94552	0.94457	0.94361	0.94264	0.94167	0.94068	0.93969	70
20	0.93969	0.93869	0.93769	0.93667	0.93565	0.93462	0.93358	69
21	0.93358	0.93253	0.93148	0.93042	0.92935	0.92827	0.92718	68
22	0.92718	0.92609	0.92499	0.92388	0.92276	0.92164	0.92050	67
23	0.92050	0.91936	0.91822	0.91706	0.91590	0.91472	0.91355	66
24	0.91355	0.91236	0.91116	0.90996	0.90875	0.90753	0.90631	65
25	0.90631	0.90507	0.90383	0.90259	0.90133	0.90007	0.89879	64
26	0.89879	0.89752	0.89623	0.89493	0.89363	0.89232	0.89101	63
27	0.89101	0.88968	0.88835	0.88701	0.88566	0.88431	0.88295	62
28	0.88295	0.88158	0.88020	0.87882	0.87743	0.87603	0.87462	61
29	0.87462	0.87321	0.87178	0.87036	0.86892	0.86748	0.86603	60
30	0.86603	0.86457	0.86310	0.86163	0.86015	0.85866	0.85717	59
31	0.85717	0.85567	0.85416	0.85264	0.85112	0.84959	0.84805	58
32	0.84805	0.84650	0.84495	0.84339	0.84182	0.84025	0.83867	57
33	0.83867	0.83708	0.83549	0.83389	0.83228	0.83066	0.82904	56
34	0.82904	0.82741	0.82577	0.82413	0.82248	0.82082	0.81915	55
35	0.81915	0.81748	0.81580	0.81412	0.81242	0.81072	0.80902	54
36	0.80902	0.80730	0.80558	0.80386	0.80212	0.80038	0.79864	53
37	0.79864	0.79688	0.79512	0.79335	0.79158	0.78980	0.78801	52
38	0.78801	0.78622	0.78442	0.78261	0.78079	0.77897	0.77715	51
39	0.77715	0.77531	0.77347	0.77162	0.76977	0.76791	0.76604	50
40	0.76604	0.76417	0.76229	0.76041	0.75851	0.75661	0.75471	49
41	0.75471	0.75280	0.75088	0.74896	0.74703	0.74509	0.74314	48
42	0.74314	0.74120	0.73924	0.73728	0.73531	0.73333	0.73135	47
43	0.73135	0.72937	0.72737	0.72537	0.72337	0.72136	0.71934	46
44	0.71934	0.71732	0.71529	0.71325	0.71121	0.70916	0.70711	45
	60′	50′	40′	30′	20′	10′	0′	Degrees
	SINE							

Degrees	TANGENT 0′	10′	20′	30′	40′	50′	60′	Degrees
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89
1	0.01746	0.02036	0.02328	0.02619	0.02910	0.03201	0.03492	88
2	0.03492	0.03783	0.04075	0.04366	0.04658	0.04949	0.05241	87
3	0.05241	0.05533	0.05824	0.06116	0.06408	0.06700	0.06993	86
4	0.06993	0.07285	0.07578	0.07870	0.08163	0.08456	0.08749	85
5	0.08749	0.09042	0.09335	0.09629	0.09923	0.10216	0.10510	84
6	0.10510	0.10805	0.11099	0.11394	0.11688	0.11983	0.12278	83
7	0.12278	0.12574	0.12869	0.13165	0.13461	0.13758	0.14054	82
8	0.14054	0.14351	0.14648	0.14945	0.15243	0.15540	0.15838	81
9	0.15838	0.16137	0.16435	0.16734	0.17033	0.17333	0.17633	80
10	0.17633	0.17933	0.18233	0.18534	0.18835	0.19136	0.19438	79
11	0.19438	0.19740	0.20042	0.20345	0.20648	0.20952	0.21256	78
12	0.21256	0.21560	0.21864	0.22169	0.22475	0.22781	0.23087	77
13	0.23087	0.23393	0.23700	0.24008	0.24316	0.24624	0.24933	76
14	0.24933	0.25242	0.25552	0.25862	0.26172	0.26483	0.26795	75
15	0.26795	0.27107	0.27419	0.27732	0.28046	0.28360	0.28675	74
16	0.28675	0.28990	0.29305	0.29621	0.29938	0.30255	0.30573	73
17	0.30573	0.30891	0.31210	0.31530	0.31850	0.32171	0.32492	72
18	0.32492	0.32814	0.33136	0.33460	0.33783	0.34108	0.34433	71
19	0.34433	0.34758	0.35085	0.35412	0.35740	0.36068	0.36397	70
20	0.36397	0.36727	0.37057	0.37388	0.37720	0.38053	0.38386	69
21	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	68
22	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	67
23	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	66
24	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	65
25	0.46631	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	64
26	0.48773	0.49134	0.49495	0.49858	0.50222	0.50587	0.50953	63
27	0.50953	0.51320	0.51688	0.52057	0.52427	0.52798	0.53171	62
28	0.53171	0.53545	0.53920	0.54296	0.54674	0.55051	0.55431	61
29	0.55431	0.55812	0.56194	0.56577	0.56962	0.57348	0.57735	60
30	0.57735	0.58124	0.58513	0.58905	0.59297	0.59691	0.60086	59
31	0.60086	0.60483	0.60881	0.61280	0.61681	0.62083	0.62487	58
32	0.62487	0.62892	0.63299	0.63707	0.64117	0.64528	0.64941	57
33	0.64941	0.65355	0.65771	0.66189	0.66608	0.67028	0.67451	56
34	0.67451	0.67875	0.68301	0.68728	0.69157	0.69588	0.70021	55
35	0.70021	0.70455	0.70891	0.71329	0.71769	0.72211	0.72654	54
36	0.72654	0.73100	0.73547	0.73996	0.74447	0.74900	0.75355	53
37	0.75355	0.75812	0.76272	0.76733	0.77196	0.77661	0.78129	52
38	0.78129	0.78598	0.79070	0.79544	0.80020	0.80498	0.80978	51
39	0.80978	0.81461	0.81946	0.82434	0.82923	0.83415	0.83910	50
40	0.83910	0.84407	0.84906	0.85408	0.85912	0.86419	0.86929	49
41	0.86929	0.87441	0.87955	0.88473	0.88992	0.89515	0.90040	48
42	0.90040	0.90569	0.91099	0.91633	0.92170	0.92709	0.93252	47
43	0.93252	0.93797	0.94345	0.94896	0.95451	0.96008	0.96569	46
44	0.96569	0.97133	0.97700	0.98270	0.98843	0.99420	1.00000	45
	60′	50′	40′	30′	20′	10′	0′	Degrees
	COTANGENT							

Degrees	COTANGENT 0′	10′	20′	30′	40′	50′	60′	
0	∞	343.77371	171.88540	114.58865	85.93979	68.75009	57.28996	89
1	57.28996	49.10388	42.96408	38.18846	34.36777	31.24158	28.63625	88
2	28.63625	26.43160	24.54176	22.90377	21.47040	20.20555	19.08114	87
3	19.08114	18.07498	17.16934	16.34986	15.60478	14.92442	14.30067	86
4	14.30067	13.72674	13.19683	12.70621	12.25051	11.82617	11.43005	85
5	11.43005	11.05943	10.71191	10.38540	10.07803	9.78817	9.51436	84
6	9.51436	9.25530	9.00983	8.77689	8.55555	8.34496	8.14435	83
7	8.14435	7.95302	7.77035	7.59575	7.42871	7.26873	7.11537	82
8	7.11537	6.96822	6.82694	6.69116	6.56055	6.43484	6.31375	81
9	6.31375	6.19703	6.08444	5.97576	5.87080	5.76937	5.67128	80
10	5.67128	5.57638	5.48451	5.39552	5.30928	5.22566	5.14455	79
11	5.14455	5.06584	4.98940	4.91516	4.84300	4.77286	4.70463	78
12	4.70463	4.63825	4.57363	4.51071	4.44942	4.38969	4.33148	77
13	4.33148	4.27471	4.21933	4.16530	4.11256	4.06107	4.01078	76
14	4.01078	3.96165	3.91364	3.86671	3.82083	3.77595	3.73205	75
15	3.73205	3.68909	3.64705	3.60588	3.56557	3.52609	3.48741	74
16	3.48741	3.44951	3.41236	3.37594	3.34023	3.30521	3.27085	73
17	3.27085	3.23714	3.20406	3.17159	3.13972	3.10842	3.07768	72
18	3.07768	3.04749	3.01783	2.98869	2.96004	2.93189	2.90421	71
19	2.90421	2.87700	2.85023	2.82391	2.79802	2.77254	2.74748	70
20	2.74748	2.72281	2.69853	2.67462	2.65109	2.62791	2.60509	69
21	2.60509	2.58261	2.56046	2.53865	2.51715	2.49597	2.47509	68
22	2.47509	2.45451	2.43422	2.41421	2.39449	2.37504	2.35585	67
23	2.35585	2.33693	2.31826	2.29984	2.28167	2.26374	2.24604	66
24	2.24604	2.22857	2.21132	2.19430	2.17749	2.16090	2.14451	65
25	2.14451	2.12832	2.11233	2.09654	2.08094	2.06553	2.05030	64
26	2.05030	2.03526	2.02039	2.00569	1.99116	1.97680	1.96261	63
27	1.96261	1.94858	1.93470	1.92098	1.90741	1.89400	1.88073	62
28	1.88073	1.86760	1.85462	1.84177	1.82907	1.81649	1.80405	61
29	1.80405	1.79174	1.77955	1.76749	1.75556	1.74375	1.73205	60
30	1.73205	1.72047	1.70901	1.69766	1.68643	1.67530	1.66428	59
31	1.66428	1.65337	1.64256	1.63185	1.62125	1.61074	1.60033	58
32	1.60033	1.59002	1.57981	1.56969	1.55966	1.54972	1.53987	57
33	1.53987	1.53010	1.52043	1.51084	1.50133	1.49190	1.48256	56
34	1.48256	1.47330	1.46411	1.45501	1.44598	1.43703	1.42815	55
35	1.42815	1 41934	1.41061	1.40195	1.39336	1.38484	1.37638	54
36	1.37638	1.36800	1.35968	1.35142	1.34323	1.33511	1.32704	53
37	1.32704	1.31904	1.31110	1.30323	1.29541	1.28764	1.27994	52
38	1.27994	1.27230	1.26471	1.25717	1.24969	1.24227	1.23490	51
39	1.23490	1.22758	1.22031	1.21310	1.20593	1.19882	1.19175	50
40	1.19175	1.18474	1.17777	1.17085	1.16398	1.15715	1.15037	49
41	1.15037	1.14363	1.13694	1.13029	1.12369	1.11713	1.11061	48
42	1.11061	1.10414	1.09770	1.09131	1.08496	1.07864	1.07237	47
43	1.07237	1.06613	1.05994	1.05378	1.04766	1.04158	1.03553	46
44	1.03553	1.02952	1.02355	1.01761	1.01170	1.00583	1.00000	45
	60′	50′	40′	30′	20′	10′	0′	Degrees
	TANGENT							

Degrees	SECANTS 0′	10′	20′	30′	40′	50′	60′	
0	1.00000	1.00001	1.00002	1.00004	1.00007	1.00011	1.00015	89
1	1.00015	1.00021	1.00027	1.00034	1.00042	1.00051	1.00061	88
2	1.00061	1.00072	1.00083	1.00095	1.00108	1.00122	1.00137	87
3	1.00137	1.00153	1.00169	1.00187	1.00205	1.00224	1.00244	86
4	1.00244	1.00265	1.00287	1.00309	1.00333	1.00357	1.00382	85
5	1.00382	1.00408	1.00435	1.00463	1.00491	1.00521	1.00551	84
6	1.00551	1.00582	1.00614	1.00647	1.00681	1.00715	1.00751	83
7	1.00751	1.00787	1.00825	1.00863	1.00902	1.00942	1.00983	82
8	1.00983	1.01024	1.01067	1.01111	1.01155	1.01200	1.01247	81
9	1.01247	1.01294	1.01342	1.01391	1.01440	1.01491	1.01543	80
10	1.01543	1.01595	1.01649	1.01703	1.01758	1.01815	1.01872	79
11	1.01872	1.01930	1.01989	1.02049	1.02110	1.02171	1.02234	78
12	1.02234	1.02298	1.02362	1.02428	1.02494	1.02562	1.02630	77
13	1.02630	1.02700	1.02770	1.02842	1.02914	1.02987	1.03061	76
14	1.03061	1.03137	1.03213	1.03290	1.03368	1.03447	1.03528	75
15	1.03528	1.03609	1.03691	1.03774	1.03858	1.03944	1.04030	74
16	1.04030	1.04117	1.04206	1.04295	1.04385	1.04477	1.04569	73
17	1.04569	1.04663	1.04757	1.04853	1.04950	1.05047	1.05146	72
18	1.05146	1.05246	1.05347	1.05449	1.05552	1.05657	1.05762	71
19	1.05762	1.05869	1.05976	1.06085	1.06195	1.06306	1.06418	70
20	1.06418	1.06531	1.06645	1.06761	1.06878	1.06995	1.07115	69
21	1.07115	1.07235	1.07356	1.07479	1.07602	1.07727	1.07853	68
22	1.07853	1.07981	1.08109	1.08239	1.08370	1.08503	1.08636	67
23	1.08636	1.08771	1.08907	1.09044	1.09183	1.09323	1.09464	66
24	1.09464	1.09606	1.09750	1.09895	1.10041	1.10189	1.10338	65
25	1.10338	1.10488	1.10640	1.10793	1.10947	1.11103	1.11260	64
26	1.11260	1.11419	1.11579	1.11740	1.11903	1.12067	1.12233	63
27	1.12233	1.12400	1.12568	1.12738	1.12910	1.13083	1.13257	62
28	1.13257	1.13433	1.13610	1.13789	1.13970	1.14152	1.14335	61
29	1.14335	1.14521	1.14707	1.14896	1.15085	1.15277	1.15470	60
30	1.15470	1.15665	1.15861	1.16059	1.16259	1.16460	1.16663	59
31	1.16663	1.16868	1.17075	1.17283	1.17493	1.17704	1.17918	58
32	1.17918	1.18133	1.18350	1.18569	1.18790	1.19012	1.19236	57
33	1.19236	1.19463	1.19691	1.19920	1.20152	1.20386	1.20622	56
34	1.20622	1.20859	1.21099	1.21341	1.21584	1.21830	1.22077	55
35	1.22077	1.22327	1.22579	1.22833	1.23089	1.23347	1.23607	54
36	1.23607	1.23869	1.24134	1.24400	1.24669	1.24940	1.25214	53
37	1.25214	1.25489	1.25767	1.26047	1.26330	1.26615	1.26902	52
38	1.26902	1.27191	1.27483	1.27778	1.28075	1.28374	1.28676	51
39	1.28676	1.28980	1.29287	1.29597	1.29909	1.30223	1.30541	50
40	1.30541	1.30861	1.31183	1.31509	1.31837	1.32168	1.32501	49
41	1.32501	1.32838	1.33177	1.33519	1.33864	1.34212	1.34563	48
42	1.34563	1.34917	1.35274	1.35634	1.35997	1.36363	1.36733	47
43	1.36733	1.37105	1.37481	1.37860	1.38242	1.38628	1.39016	46
44	1.39016	1.39409	1.39804	1.40203	1.40606	1.41012	1.41421	45
	60′	50′	40′	30′	20′	10′	0′	Degrees
	COSECANTS							

COSECANTS

Degrees	0′	10′	20′	30′	40′	50′	60′	
0	∞	343.77516	171.88831	114.59301	85.94561	68.75736	57.29869	89
1	57.29869	49.11406	42.97571	38.20155	34.38232	31.25758	28.65371	88
2	28.65371	26.45051	24.56212	22.92559	21.49368	20.23028	19.10732	87
3	19.10732	18.10262	17.19843	16.38041	15.63679	14.95788	14.33559	86
4	14.33559	13.76312	13.23472	12.74550	12.29125	11.86837	11.47371	85
5	11.47371	11.10455	10.75849	10.43343	10.12752	9.83912	9.56677	84
6	9.56677	9.30917	9.06515	8.83367	8.61379	8.46466	8.20551	83
7	8.20551	8.01565	7.83443	7.66130	7.49571	7.33719	7.18530	82
8	7.18530	7.03962	6.89979	6.76547	6.63633	6.51208	6.39245	81
9	6.39245	6.27719	6.16607	6.05886	5.95536	5.85539	5.75877	80
10	5.75877	5.66533	5.57493	5.48740	5.40263	5.32049	5.24084	79
11	5.24084	5.16359	5.08863	5.01585	4.94517	4.87649	4.80973	78
12	4.80973	4.74482	4.68167	4.62023	4.56041	4.50216	4.44541	77
13	4.44541	4.39012	4.33622	4.28366	4.23239	4.18238	4.13357	76
14	4.13357	4.08591	4.03938	3.99393	3.94952	3.90613	3.86370	75
15	3.86370	3.82223	3.78166	3.74198	3.70315	3.66515	3.62796	74
16	3.62796	3.59154	3.55587	3.52094	3.48671	3.45317	3.42030	73
17	3.42030	3.38808	3.35649	3.32551	3.29512	3.26531	3.23607	72
18	3.23607	3.20737	3.17920	3.15155	3.12440	3.09774	3.07155	71
19	3.07155	3.04584	3.02057	2.99574	2 97135	2.94737	2.92380	70
20	2.92380	2.90063	2.87785	2.85545	2.83342	2.81175	2.79043	69
21	2.79043	2.76945	2.74881	2.72850	2.70851	2.68884	2.66947	68
22	2.66947	2.65040	2.63162	2.61313	2.59491	2.57698	2.55930	67
23	2.55930	2.54190	2.52474	2.50784	2.49119	2.47477	2.45859	66
24	2.45859	2.44264	2.42692	2.41142	2.39614	2.38107	2.36620	65
25	2.36620	2.35154	2.33708	2.32282	2.30875	2.29487	2.28117	64
26	2.28117	2.26766	2.25432	2.24116	2.22817	2.21535	2.20269	63
27	2.20269	2.19019	2.17786	2.16568	2.15366	2.14178	2.13005	62
28	2.13005	2.11847	2.10704	2.09574	2.08458	2.07356	2.06267	61
29	2.06267	2.05191	2.04128	2.03077	2.02039	2.01014	2.00000	60
30	2.00000	1.98998	1.98008	1.97029	1.96062	1.95106	1.94160	59
31	1.94160	1.93226	1.92302	1.91388	1.90485	1.89591	1.88709	58
32	1.88708	1.87834	1.86990	1.86116	1.85271	1.84435	1.83608	57
33	1.83608	1.82790	1.81981	1.81180	1.80388	1.79604	1.78829	56
34	1.78829	1.78062	1.77303	1 76552	1.75808	1.75073	1.74345	55
35	1.74345	1.73624	1.72911	1.72205	1.71506	1.70815	1.70130	54
36	1.70130	1.69452	1.68782	1.68117	1.67460	1.66809	1.66164	53
37	1.66164	1.65526	1.64894	1.64268	1.63648	1.63035	1.62427	52
38	1.62427	1.61825	1.61229	1.60639	1.60054	1.59475	1.58902	51
39	1.58902	1.58333	1.57771	1.57213	1.56661	1.56114	1.55572	50
40	1.55572	1.55036	1.54504	1.53977	1.53455	1.52938	1.52425	49
41	1.52425	1.51918	1.51415	1.50916	1.50422	1.49933	1.49448	48
42	1.49448	1.48967	1.48491	1.48019	1.47551	1.47087	1.46628	47
43	1.46628	1.46173	1.45721	1.45274	1.44831	1.44391	1.43956	46
44	1.43956	1.43524	1.43096	1.42672	1.42251	1.41835	1.41421	45
	60′	50′	40′	30′	20′	10′	0′	Degrees

SECANTS

YA 06861